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I. *Presidential Address.* *By* Dr. Alexander Russell.

Delivered October 27, 1922.

In this address I shall first give my views of some recent developments in physical science. Not because I am specially qualified to talk on these subjects—some of you have studied them far more deeply than I have—but because it may be of value as showing how the average scientific man is struggling to get a sure foundation for his knowledge.

New Physical Laws.—Since our Society was founded 48 years ago, several new physical laws have been discovered. Newton pointed out that it redounds to the credit of theory to have as few fundamental principles as possible. Physicists, therefore, have expended almost as much thought in trying to deduce these laws from primary causes as in discovering new practical applications for them. There is nothing that gives an investigator greater pleasure than discovering that some apparently complicated phenomenon can be rigorously deduced from fundamental principles. As a rule, however, this discovery requires ability of the highest order. For example, when Snow Harris found that two spheres electrified with like electricities sometimes attracted and sometimes repelled one another, it needed the genius of a William Thomson to prove that Harris's phenomena followed rigorously from the laws of electricity enunciated by Coulomb. The physical explanation of some of the phenomena discovered in recent years still baffles us, but there can be little doubt, judging by the ability and perseverance of the present generation of physicists, that their secrets will soon be wrested from Nature.

The main functions of the Physical Society are to disseminate the knowledge of new discoveries and examine critically new theories from every point of view. A look through our Proceedings and through the Reports written by eminent physicists which have been issued from time to time will show how well it has fulfilled its functions.

Quantum Theory.—It was no easy matter for Planck's Quantum Theory to make headway 20 years ago. The Society, therefore, invited Mr. J. H. Jeans, Sec. R.S., to write a Report on Radiation and the Quantum Theory. This Report has been of the greatest value and has been helpful to workers all over the world. Planck's constant is now considered to be one of the fundamental constants in Nature. The apparent difficulty that energy is radiated in quanta will probably disappear when the atomic mechanism of matter is more fully understood. Physicists are looking forward with great interest to the appearance of the new edition of this report.

Relativity.—Although the theory of Relativity has been widely discussed for several years, little progress has been made in presenting the theory in a form readily intelligible by the public. Even the most elementary explanations of the subject finish up by referring to a four dimensional continuum or a four-dimensional manifold. These words convey a meaning to nobody but a mathematician. A simple explanation without mathematical conceptions seems to be impossible. The Report on the "Relativity Theory of Gravitation" which Professor Eddington wrote for the Society in 1918 is still one of the best introductions to the practical side of this difficult theory. The clear way in which he states the crucial phenomena by means of which Einstein's gravitational theory will be adjudged can be appreciated by everyone. It is well known that a charged body in motion behaves as if it had an additional mass, the reason being that the moving charge generates a magnetic field in the surrounding region. This increases the energy of the moving system. Hence it is equivalent to the effect produced by an increase in the mass of the body. This can be readily understood. But on the theory of relativity the mass, whether it have an electric charge or not, varies with the speed. A certain amount of experimental evidence can be deduced in support of this conclusion, but its foundations certainly want strengthening.

Realities.—In his address to Section A of the British Association this year, Professor Hardy said that the old-fashioned geometry of Euclid and the space-times of Einstein and Minkowski were " realities." On the other hand he pointed out that no philosopher or physicist had ever put forward any convincing account of what " reality " was. I hope it will be a long time before any physicist wastes time in trying to define physical " reality." Physicists only claim to know what they have learned from experience.

The Position of the Physicist.—In the opening sentence of Fourier's " Theory of Heat," our position is clearly stated :—

" Primary causes are unknown to us but are subject to simple and constant laws which may be discovered by experiment and the study of which is the object of Natural Philosophy." Thomson and Tait prefixed this sentence in the original French to their Treatise on Natural Philosophy. They point out that most useful and interesting results follow by mathematical processes from known physical laws. This seems to contradict Professor Hardy's statement that it is impossible to produce by mathematical reasoning any proposition at all concerning the physical world. I think that every one who sees the progress that physical science, helped by mathematics, is making at the present time must admit that it is hourly improving man's mastery over Nature.

Series in Line Spectra.—This year a very valuable and exhaustive report on " Series in Line Spectra," by Professor Fowler, was published by the Society. To physicists the subject is of absorbing interest. The discovery by Balmer in 1885 of the law according to which the hydrogen lines arrange themselves in a spectrum marks an epoch in the history of spectrum analysis. The simplicity and accuracy of the law was a great inducement to workers to search for possible physical causes, and most satisfactory progress has been made.

The great importance of the study of spectra lies in the fact that it must give us valuable information about the constitution of matter. This was early recognised. As far back as 1867, Sir William Thomson (Kelvin) suggested that the two principal lines in the sodium spectrum might be due to the fact that the sodium atom had two fundamental modes of vibration whose periodic times were the same as that of the lines they produced. At that time he was elaborating a vortex theory of the ether and so he suggested that the sodium atom consisted of two approximately equal vortex rings passing through one another like links in a chain. It is difficult to see how this assumption could be elaborated so as to explain the 65 lines given for Sodium in Fowler's Report. Rutherford's nuclear theory, however, will probably furnish in the future far simpler and more satisfactory explanations.

The Electron and the Atom.—The discovery that external magnetic and electrostatic fields could alter the position of the lines in the spectrum (the Zeeman and

Stark effects) is strong evidence that electricity plays a leading part in the atomic constitution of matter. During the last century the theory of electricity was as unsatisfactory as the experimental verification of the results obtained from a few fundamental assumptions was satisfactory. Consider, for instance, the Paper on the electrification of two spherical conductors read by Poisson in 1812 to the French Academy. He made the supposition that electricity consisted of two fluids. Every molecule of one fluid attracted every molecule of the other fluid with a force inversely proportional to the square of the distance between them. It also repelled every molecule of a like fluid according to the same law. By rigorous and elementary mathematical reasoning he showed how to find the capacities of the two spheres in the presence of each other, and for particular cases he worked out their accurate numerical values. Although he supposed the layer of electricity on each sphere to be practically infinitely thin, yet he found formulæ to give the density, and consequently the electric stress, at points on the two surfaces. The accuracy of his results probably still far exceeds the limits of accuracy of the most sensitive present-day apparatus.

The great advance that has been made in recent years is the discovery of the electron by Sir J. J. Thomson. The importance of this discovery can hardly be over-estimated. The electron undoubtedly plays a leading part in the constitution of the atom. It gave us our first insight into what is happening in the ultra-microscopic region of matter. Judging by the important results obtained by Rutherford's nuclear theory of the constitution of matter, it must be given the leading place amongst atomic theories. Accepting this hypothesis and Planck's theory of quanta, Bohr has shown that the position of the lines in the hydrogen spectrum can be computed with an accuracy as great as that of the most refined experimental measurements. It is difficult to believe that his theory is not substantially accurate. His theory also applies to ionised helium, where we have a single electron rotating round a nucleus of four times the mass of the hydrogen nucleus. In the Guthrie lecture which Professor Bohr gave us this year it was indicated how the theory could be expanded so as to explain the spectra of more complex elements. According to Bohr and Sommerfeld, the nucleus is surrounded by various groups of orbits. For all orbits of one group the energy of the electrons is substantially the same, but as some of the orbits are elliptical the energy given up when an electron jumps from one orbit to another may not be an exact quantum. The frequency of the radiation, therefore, may have two or more values, differing slightly from one another. This would explain how a single spectrum line is sometimes doubled or trebled, giving us what Sommerfeld calls the fine structure of the lines. Dr. Silberstein has recently shown that if we make the assumption that the mutual forces between the electrons of a neutral helium atom are negligibly small, then by the theory of quanta we can compute the positions of at least 40 lines in the helium spectrum with the highest accuracy. It is difficult to see any reason why such accurate results should follow from an erroneous assumption. W. M. Hicks has also given analytical relations connecting the lines in the spectrum of hydrogen. There are now so many relations to explain that any theory which explains them all will be very convincing. It may be possible to do without planetary electrons by the use of an atomic model such as that devised by Professor Whittaker, but many of us believe that Bohr's theory has a substantial foundation.

It has to be remembered that no satisfactory explanation has yet been given of the stability of molecules in terms of the mutual actions taking place between their atoms. Lodge suggests that we should abandon the idea of electrical action between atoms producing chemical force, and substitute for it the forces produced by the interlacing magnetic fields produced by the rapidly revolving electrons. It is possible that chemical affinity might be explained in this way. It is advisable, therefore, for physicists to keep an open mind and see how far the various hypotheses that have been made can explain physical phenomena. They at least suggest new lines of research.

The Moseley classification of the elements according to the electric charges on the nuclei of their atoms has produced a wonderful simplification in theory. In his series of ordinal numbers for the chemical elements ranging from hydrogen to uranium (1 to 92) there are now only three numbers missing—namely, 43, 61 and 75. It is probable that the elements corresponding to these numbers will soon be discovered. The discovery of isotopes—that is, atoms which have the same nuclear charge, but which have different masses—will be a great help in furthering our knowledge. They prove that the physical and chemical properties of the atom depend only on its nuclear charge, and are independent of its mass.

Text Books.—The time has now come when the text books of physics in everyday use will have to be largely re-written. Recent advances in physics are but feebly reflected in some of them. A few books, like Newton's *Principia Mathematica*, are written for all time ; but in the light of recent researches there is a large field of work open to writers in reorganising our knowledge. There is always a large amount of inertia to be overcome before any change can be made in the orthodox ways of teaching science, but it is of national importance that the community should recognise the potential value of recent discoveries. Rutherford has induced artificial radio-activity in boron, nitrogen, fluorine, sodium, aluminium and phosphorus by means of the high-speed helium nucleus. He has proved that the hydrogen atoms leave the nucleus with more energy than that possessed by the helium projectile which caused the emission. Theoretically, therefore, it is possible to obtain some of the atomic energy stored in matter. Manufacturers should recognise that knowledge of the constitution of matter obtained by research in pure physics will probably be of the greatest practical value to them.

Applied Research.—One of the great differences between pure and applied research is that the former is generally concerned with the simple elements, whilst the latter has to study the materials used in everyday work. Consider rubber, and the composite materials used for insulating electric cables. We desire to know their electric strength, their inductivity, their dielectric losses at high and low frequencies, their capacity for moisture, their ignitability, their insulating, their mechanical and their thermal properties, and lastly their life under normal conditions of working. With simple elements we can generally make the assumption that their physical properties remain constant. It is not so with some of the materials used in commerce.

In the old days, a good buyer with the aid of a pocket-knife and a careful visual examination was able to purchase material which was generally found to be satisfactory by his firm. Now the manufacturers have combined together, and with the help of the Department of Scientific and Industrial Research, of the National Physical Laboratory, and the leading research laboratories, are carrying out elaborate researches to find out which of the physical qualities of a material are of the greatest importance in practice. They are also devising methods to enable these qualities to be easily and accurately measured. The material may offer great resistance to the passage of a disruptive spark through it, but engineers have to be satisfied that this electric strength is a permanent quality of the material. Some cables when buried in the ground deteriorate rapidly. It is necessary, therefore, to make costly and lengthy experiments to investigate these and similar phenomena.

Methods of Duality.—It is found sometimes that the experimental determination of a problem in one branch of science enables us to solve the analogous problem in another. Kelvin always laid the greatest stress on these analogies. He pointed out, for example, the analogy between the motion of a vortex filament in a perfect fluid and the magnetic field round a circular current. The expression for the total kinetic energy of the vortex ring and the moving fluid can be written down at once from the expression for the electro-magnetic energy stored up in the field round the circular current. Similarly, the electric resistance between two spheres in a homogeneous medium can be written down when we know the electrostatic capacity between them.

These analogies are of practical importance. For instance, when we find out experimentally the electrostatic capacity between the cores and the sheath of a polycore cable we can deduce the thermal resistance to a steady flow of heat from the cores to the sheath in terms of the thermal conductivity of the insulating material. Hence by measuring the thermal resistance we get the true value of the thermal conductivity of the material. I do not know whether this method has yet been adopted. In the ordinary experimental method a mathematical formula is used, but as this formula neglects, amongst other things, the twist of the cores round the central axis of the cable it is affected by errors the magnitude of which are unknown; hence the combined electrostatic and thermal method is preferable. Similarly, if the insulating material of the cable obeys Ohm's law its insulation resistance per mile can be found either from the electrostatic or from the thermal measurements.

As a help in discovering new theorems and applications the methods of duality are of great value in physics. Some of the analogies are obvious. Consider, for example, the method of images and the phenomena of refraction in light, sound, magnetism, electrostatics, and electrokinetics. In some cases the analogies have to be laboriously constructed. In electrical science, duality theorems can be deduced by interchanging resistance and conductance, current and pressure, inductance and capacity, star and mesh, electrostatic and electromagnetic instruments, and dynamos and motors. In many cases these theorems are as rigorous as the theorems in duality deduced in geometry.

As an example of duality on a large scale we can take Mr. George Constantinesco's method of transmitting power by transmitting vibrations through water contained in pipes. The rapid development of this method followed at once when the analogy between the vibrations of liquids in pipes and alternating currents in wires was seen. Practically the whole theory of the transmission of power by alternating currents together with that of all the accessory generators, motors, transformers and condensers can be directly applied both to give a working theory of the transmission of power by hydraulic waves and to develop a system of hydraulic machines exactly analogous to those used in electrical work. We have, for instance, hydraulic generators and motors of the single and polyphase type. We also have both synchronous and asynchronous machines.

The Electrical Units.—Pure scientific research has probably been of greater help to the electrical than to any other industry. The invaluable work done by the British Association Committee, both in defining our electrical units and in devising methods of determining them in terms of the centimetre, the gramme and the second is described in Gray's " Absolute Measurements." The electrical industry rightly reveres the memory of Maxwell, Kelvin and Rayleigh. Their thanks are also due to our secretary, Mr. F. E. Smith, whose skill and physical insight enabled him to measure the electrical units with an accuracy which is at present in excess of all commercial requirements. I agree with him in thinking that all electrical students should be taught how to measure resistance in electromagnetic measure by means of a rotating coil. They can also measure current in absolute measure by a tangent galvanometer or by weighing the attraction between coils carrying currents. Hence by Ohm's law they can determine the unit of E.M.F. in absolute electromagnetic measure. I should suggest also that they weigh the electrostatic attraction between two spherical conductors maintained at a high potential difference. They can then easily compute the E.M.F. in electrostatic measure, and measuring it in electromagnetic measure by a voltmeter, they find that the electrostatic unit of pressure is 300 volts, and the ratio of the electrostatic to the absolute electromagnetic unit of pressure is 3×10^{10}, which gives the velocity of light in centimetres per second.

Precipitation of Vapours.—A very useful field of research is in connection with the electrical precipitation of vapours. In many industries smoke vapours and dust laden gases are discharged into the atmosphere. In some cases they carry off valuable by-products. The commercial efficiency of the works is thereby appreciably lowered. In all cases the smoke and vapours constitute a nuisance, and in some cases they constitute a danger. The nuisance may be mitigated and the danger removed by electrical precipitation. In this process charged electrodes which exercise forces of attraction and repulsion on neighbouring particles are used. The electrodes are so shaped that one promotes discharge and the other makes it a minimum. In works the deposit is usually removed from the electrodes by tapping them every three or four hours with hammers actuated by electric motors. The insulating framework is also periodically vibrated. In the Cottrell process when the flow of gases is about 170 cubic ft. per second, an input of 2 kw. at a pressure ranging from 25 to 100 thousand volts is required. Precipitation is also of great value in smelting works and in cement plants. In the former case the recovery of the waste metals—copper, silver, or tin, for instance—is the main object in view. In America the Cottrell plant for copper smelters treats 130,000 cubic ft. of gases per second and recovers about 5 million pounds of copper per annum. The method can also be used to precipitate coal smoke, where unfortunately the by-product is of little or no value. As gas or electricity can now always be used to heat buildings in towns, the time has come when coal consumers should be penalised. When coal is burnt some form of soot precipitation plant should be made compulsory. It is better policy to prevent the causes of fogs than to invent remedies for the fogs themselves.

Acoustics.—There has recently been considerable discussion as to the best method of securing good acoustical properties in a large room. It is agreed that good conditions for hearing can only be secured by eliminating reverberation due to reflexion from the boundaries of the room. If we knew more about the acoustical properties of matter it should not be difficult to prevent this reverberation without the necessity of altering the design of the room or of appreciably increasing the cost. In this connection Sir Arthur Schuster has pointed out that experimental investigation of the acoustical properties of materials with regard to absorption, scattering and rate of transmission is still required. A somewhat analogous and even more pressing problem which has to be solved is the construction of noiseless and vibrationless sub-stations for power and railway supply in cities and large towns. The converse problem of making a sound-proof room for use in telephonic research also needs investigation. The Wallace Sabine laboratory of acoustics in America has done excellent work, but much further work remains to be done.

Radio Communication.—Radio-communication is now a great and popular branch of applied science. There can be no doubt that broadcasting when it becomes general will have a great influence on national development. The fascination of the art is attracting many able young men to take up applied electricity as a career. The labours of physicists have placed this branch of science on a sound foundation. When once Professor Fleming realised the mode of action of the electrons in the Edison effect, the thermionic valve came into existence and subsequent developments followed with almost bewildering rapidity.

The main difficulty yet to be overcome is how to eliminate the effects of atmospherics from radiophones. There are two ways of minimising these effects—namely, by increasing the sharpness of the tuning, and by directional reception—that is, by taking the signals from the aerial at the best point. In order that the signals may be heard in any kind of weather it is necessary to erect the sending stations on the most suitable site and make them powerful. As unanimity of control has been secured we can hope that the mistakes which have been made in America, where broadcasting is in rather a chaotic condition, will be avoided. It is estimated that in America there are about two million radio-receiving sets in use. In the neighbourhood of New York there are no less than 20 broadcasting stations, each using $1\frac{1}{2}$ kw. and emitting a wave-length of 360 metres.

It will be of interest to see whether the other method of broadcasting, namely, through the mains of the electric lighting supply companies, will make progress. It is certainly scientifically feasible.

Electrons in Engineering.—Richardson's researches proving that electrons are emitted by hot metal by a process similar in all respects to evaporation were invaluable in the development of thermionic devices. At first sight these investigations appear to have little to do with heavy electrical engineering. Leblanc, however, has recently shown how the large amount of power required by a high speed electric train can be communicated to it without rubbing contacts by magnetic induction. He proposes to use alternating current having a frequency of 20,000. This is carried over the track by a series of tubular condensers adjusted to resonance. The current in the locomotive circuit is converted to low frequency by means of valve transformers which operate ordinary induction motors as in other alternating-current railway systems. Besides the advantage of doing away with rubbing contacts, it has the advantage that any interference produced in neighbouring telegraph and telephone circuits can easily be remedied by means of condensers. Leblanc's suggestions show how recondite mathematical theorems and apparently unpractical physical investigations can be combined to devise a new and promising system for heavy electric traction.

Individual versus Collective Effort.—Progress in new branches of industry needs, in the first place, individual effort. Individual workers with simple apparatus and a whole-hearted enthusiasm are more likely to obtain valuable results than the best organised collective effort with costly and elaborate plant. Collective effort is only advisable when it is almost certain that valuable results will be obtained. Brilliant discoveries are usually made in out-of-the-way fields where few ever penetrate and where the prospects of a successful issue are too remote to justify a collective effort.

Development Departments.—It has to be remembered, however, that when a new discovery is made which may have commercial applications, then, as a rule, it can be developed far better by the Development Department of a large industrial undertaking than by the individual inventor. I think, therefore, that a closer co-operation between physicists and industrial engineers is highly desirable and will be to their mutual advantage. In pre-war days the discoveries of British scientists were often first commercially exploited in Germany. For example, Sir James Dewar's vacuum flask for preventing bodies from gaining or losing heat was first made on a commercial scale in Germany. This was due to the more thorough organisation of the Development Departments in German works. I am glad to see that there are signs that the great importance of the work done by these departments is being recognised in this country.

The Physical Society.—It is interesting to remember that the Physical Society of London was founded in the same year that Graham Bell began his researches in telephony. The invention of the telephone was a boon to humanity. It shows how pure research can benefit the world and should encourage physicists to develop the practical side of their discoveries. The first Paper to the Society was read on March 21, 1874, by Professor J. A. Fleming, whom every physicist and engineer delights to honour. It was entitled " On the New Contact Theory of the Galvanic Cell."

We have recently had to lament the death of two of our Fellows, Professor F. T. Trouton and F. W. Sanderson. Professor Trouton, who has served on the Council of this Society, was one of Fitzgerald's most distinguished pupils. He did excellent work on Hertzian waves and, in conjunction with Professor Rankine, carried out an important experiment to try and detect the Fitzgerald shrinkage in the theory of relativity. Amongst other work, he discovered the connexion between latent heat and molecular weight.

F. W. Sanderson did invaluable work in promoting pure and applied science teaching in this country. For 30 years he was headmaster of Oundle School and carried out many novel and successful experiments in education. His greatest success lay in interesting practically every boy in his school in some branch of study. By the deaths of Sanderson, of Oundle, and McClure, of Mill Hill, the country has lost two great educational experimentalists. Their success proves the wisdom of allowing teachers to develop their own individual methods.

Conclusion.—I should like to emphasise that there has seldom, if ever, been such an interesting period in the history of physical science as the present time. It was a great shock to scientists to find that the velocity of light was independent of the motion of the observer. Some explained it by saying that the earth dragged the ether along with it, but this is contradicted by the phenomenon of the aberration of light. Fitzgerald and Lorentz suggested independently the contraction hypothesis which gives a rational explanation, but which is very difficult to verify by experiment. The relativity theory does not explain the nature of the phenomenon, but it indicates that the hypothesis of the ether is not a necessity. So many exciting problems have arisen in recent years that Rutherford's experiments showing the transmutation of gases have not received the attention they deserve. These experiments mark a new epoch in the history of science, and in the future they may have marvellous commercial developments. The further our knowledge of physical science extends the more closely does it affect us in our normal everyday mental and physical life. A country that neglects the study of physical science cannot take a leading place amongst civilised nations. It is our duty to the human race as well as to our own country to do our utmost to extend the knowledge of the universe in which we live.

II. *The Problem of Two Electrified Spheres.* *By* ALEXANDER RUSSELL, *M.A., D.Sc., Principal of Faraday House.*

RECEIVED SEPTEMBER 21, 1922.

ABSTRACT.

It is shown that Poisson's method can be readily applied to get the complete solution of the problem of two electrified spherical conductors. The case of a spherical condenser is first considered and simplified methods of computing its capacity are given. Simple expressions for the potentials at the two inverse points due to the charges on each of the spheres are found. Poisson's method is then used to show that the capacity coefficients for two spheres external to one another can be very simply expressed in terms of the capacities of certain spherical condensers and can thus be readily computed in all cases. Formulæ for the attraction and repulsion of two electrified spheres are given. When the charges are of like sign the electrostatic energy of the system has a maximum value when the spheres are at a given distance apart. Similarly, when the potentials are of like sign and maintained constant, the electrostatic energy of the system has a minimum value for a definite distance between them. The particular case when the spheres are close together is considered in detail. The exactly analogous problems in thermal and electric conduction are also considered, and it is shown how their solutions can be written down at once. Finally, the formula for the maximum value of the electric stress in the medium surrounding the spheres is given and its practical importance is pointed out. For equal spherical electrodes at a fixed distance apart there is a definite size of radius which makes the disruptive potential difference a maximum. The distance between them where the attraction has a maximum value is also found.

THE problem of two electrified spherical conductors has attracted the study of physicists for over a hundred years. From the point of view of the mathematician it may be considered that the complete solution has been obtained. The solutions, however, are usually given in the form of series which are in general very laborious to compute numerically, and hence they are rarely used by the physicist. The author has therefore thought it advisable to give what he considers to be the simplest solution of the problem, and to show by numerical examples that in practically every case the numerical solution can be readily obtained. The methods of computing the capacity of a spherical condenser are first considered. It is then shown how the self and mutual capacity coefficients of external spheres can always be written down by simple formulæ in terms of the capacities of certain spherical condensers, and can thus be readily computed. Finally their attractions and repulsions, the maximum value of the potential gradient at their surfaces and the value of the potentials at all points are given.

In calculating the capacity of a spherical condenser* the author formerly used Kelvin's method of images. Difficulties arose as in some cases the Kelvin points were at infinity. In what follows he gives a modification of the method which Poisson† gave 112 years ago. He now thinks that this method is the simplest that has so far been devised.

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* Proc. Roy. Soc., Vol. 94, p. 206.

† Memoires de l'Institut (1812).

LEMMA.

Consider an electrified spherical conductor of radius a with its centre at O (Fig. 1). The charge q on it is distributed over its surface. Let the potential due to this charge at a point P distant x from O be denoted by $\varphi(x/a)$. Then if $OP \,.\, OQ = a^2$, we have

$$\varphi\left(\frac{x}{a}\right)=\Sigma\frac{\delta q}{AP}=\frac{a}{OP}\Sigma\frac{\delta q}{AQ}=\frac{a}{x}\varphi\left(\frac{a}{x}\right). \qquad (1)$$

This is true, whether x is greater or less than a. Notice also that $\varphi(0)=\frac{q}{a}$.

THE CAPACITY OF A SPHERICAL CONDENSER.

Let A (Fig. 2) be the centre of a spherical cavity in a conducting solid, and let B be the centre of a spherical conductor in this cavity. Let a and b be the radii of the two spheres, and let c be the distance AB between their centres. Let the inner sphere have a charge q. There will be a charge $-q$ over the surface of the cavity.

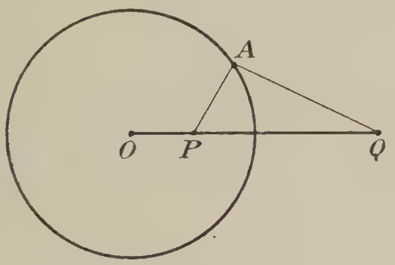

FIG. 1.

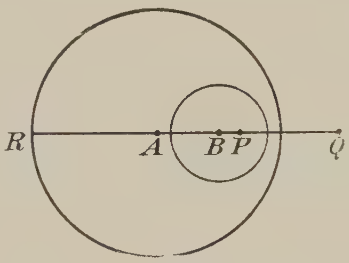

FIG. 2.

The potential at a point in the outer conductor at a great distance away will be zero, for its distances from the charges $+q$ and $-q$ will be practically the same. Hence every point on the outer conductor must be at zero potential. We shall suppose that the inner conductor is at potential V. Let P (Fig. 2) be any point on the line $RABQ$, inside the sphere B, and let $AP=x$. Let $f_1(x/a)$ be the potential at P produced by the charge $-q$, and let $f_2\{(x-c)/b\}$ be the potential at the same point produced by the charge $+q$. Since the outer conductor is at potential zero, we have for the potential at a point Q in it, on AB produced

$$f_1\left(\frac{AQ}{a}\right)+f_2\left(\frac{BQ}{b}\right)=0$$

Hence if $AQ=\xi$ we have by (1)

$$\frac{a}{\xi}f_1\left(\frac{a}{\xi}\right)+\frac{b}{\xi-c}f_2\left(\frac{b}{\xi-c}\right)=0 \qquad (2)$$

where ξ lies outside the limits $+a$ and $-a$.

We have also at every point P on RQ inside the sphere B

$$f_1\left(\frac{x}{a}\right)+f_2\left(\frac{x-c}{b}\right)=V \qquad (3)$$

where x lies in value between $c-b$ and $c+b$.

Now let $\frac{x}{a}=\frac{a}{\xi}$. This is always possible since a must be greater than $b+c$.

Hence eliminating $f_1\left(\frac{x}{a}\right)$ between (2) and (3) we get

$$f_2\left(\frac{x-c}{b}\right)-\frac{ab}{a^2-cx}f_2\left(\frac{bx}{a^2-cx}\right)=V. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we write

$$\frac{x_n-c}{x_{n-1}}=\frac{b^2}{a^2-cx_{n-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we deduce by successive applications of (4) that

$$f_2\left(\frac{x_1-c}{b}\right)=V\left\{1+\frac{ab}{a^2-cx_1}+\frac{a^2b^2}{(a^2-cx_1)(a^2-cx_2)}+\ .\ .\ .\right\}. \quad . \quad . \quad . \quad (6)$$

The formulæ (6) and (5) enable us to compute the potential due to the charge $+q$ at all points on the diameter of the inner sphere which passes through A and B.

It is convenient to express a, b and c in terms of certain functions α, β and r, which may be defined* by the equations

$$a=r/\sinh\alpha,\ \ b=r/\sinh\beta,\ \ c=r\sinh(\beta-\alpha)/(\sinh\alpha\sinh\beta) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It can be shown that r is the radius of the sphere which cuts the two spheres A and B orthogonally, and which has its centre on AB produced. If we denote $\beta-\alpha$ by ω, we have

$$\cosh\alpha=\frac{a^2+c^2-b^2}{2ac},\ \cosh\beta=\frac{a^2-b^2-c^2}{2bc},$$

$$\cosh\omega=\frac{a^2+b^2-c^2}{2ab},\ \text{and}\ \sinh\omega=\frac{cr}{ab}.$$

Hence also $c=a\cosh\alpha-b\cosh\beta=a\varepsilon^{\alpha}-b\varepsilon^{\beta}=a\varepsilon^{-\alpha}-b\varepsilon^{-\beta}$ . . . . (8)

The potential at B (Fig. 2) due to the charge q is obviously q/b, and this equals KV/b, where K is the capacity of the spherical condenser. Hence, putting $x_1=c$, and noting that $f_2(o)$ is the potential at B, we get

$$\frac{K}{b}=1+\frac{ab}{a^2-c^2}+\frac{a^2b^2}{(a^2-c^2)\,(a^2-cx_2)}+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where x_2, x_3, . . . are found successively by (5).

Now
$$\frac{ab}{a^2-cx_1}=\frac{\sinh\beta}{\sinh(\beta+\omega)}.$$

and
$$\frac{ab}{a^2-cx_2}=\frac{\sinh(\beta+\omega)}{\sinh(\beta+2\omega)},\ \text{etc.}$$

Hence
$$K=\sum_0^\infty\frac{r}{\sinh(\beta+n\omega)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This agrees with formula (5), given in the Proc. Roy. Soc. A., Vol. XCIV., p. 209, which was obtained by Kelvin's method of images. We see that (10) can be written in the form

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* Russell, Alternating Currents, Vol. 1, p. 166.

$$K=\frac{r}{\sinh\beta}+K_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Where K_1 is the capacity of a spherical condenser for which the inner radius b_1 equals $r/\sinh(\beta+\omega)$, the outer radius a_1 equals $r/\sinh(\beta+\omega-\omega)$ and the distance between their centres c_1 equals $\frac{a_1b_1}{r}\sinh\omega$.

We shall denote the capacity of a spherical condenser by $K(a, b, c)$. Hence

$$K(a,b,c)=b+K\left(b, \frac{ab^2}{a^2-c^2}, \frac{cb^2}{a^2-c^2}\right)=b+\frac{b^2}{a^2-c^2}K\left(\frac{a^2-c^2}{b}, a, c\right) \quad . \quad . \quad . \quad . \quad (12)$$

for $K(ma, mb, mc)=mK(a, b, c)$.

As the condenser on the right-hand side of (12) is always more nearly concentric than the one on the left-hand side, (12) is very useful for computing purposes.

The author has previously shown (*l.c. ante*) that

$$K(a, b, c)=\frac{ab}{a-b}\left\{1+A_2\frac{c^2}{(a-b)^2}+A_2A_4\frac{c^4}{(a-b)^4}+\,.\,.\right\}$$

where $A_2=\frac{ab}{a^2+b^2+ab}$, and A_2A_4 has its maximum value $\frac{1}{5}$ when $a=b$.

Hence, when $c^4/\{5(a-b)^4\}$ can be neglected compared with unity, we have

$$K(a, b, c)=\frac{ab}{a-b}\left\{1+\frac{ab}{a^2+b^2+ab}\cdot\frac{c^2}{(a-b)^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

with high accuracy.

When $c/(a-b)$ is large, it is better to use the formula (*l.c. ante*)

$$K(a, b, c)=\frac{b}{2}+\frac{r}{2\omega}\log_\epsilon\frac{\cosh\beta+1}{\cosh\beta-1}+\frac{r\omega}{12}\;\frac{\cosh\beta}{\sinh^2\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Writing $2s=a+b+c$, we get

$$cr=2[s(a-s)(s-b)(s-c)]^{\frac{1}{2}}$$

$\cosh(\omega/2)=\{s(s-c)/ab\}^{\frac{1}{2}}$; $\sinh(\omega/2)=\{(a-s)(s-b)/ab\}^{\frac{1}{2}}$; $\omega/2=\log_\epsilon\{\cosh(\omega/2)+\sinh(\omega/2)\}$; and $\cosh(\beta/2)=\{(s-b)(s-c)/bc\}^{\frac{1}{2}}$.

The formulæ give the capacities in electrostatic units (centimetres). To find the capacities in microfarads we divide by 900,000.

NUMERICAL EXAMPLES.

Using (12), we get

$$K(10, 1, 6)=1+\frac{1}{64}K(64, 10, 6)$$

$$=1+\frac{1}{32}K(32, 5, 3)$$

Similarly

$$K(32, 5, 3)=5+\frac{5}{203}K(203, 32, 3)$$

We see by (13) that the error made in computing $K(203, 32, 3)$ as if it were the concentric spherical condenser $K(203, 32, 0)$ is very small. Writing, therefore,

$$K(203, 32, 3)=\frac{203 \cdot 32}{203-32}$$

and, substituting, we get

$$K(10, 1, 6)=1{\cdot}185490 \ldots$$

The true value is $1{\cdot}185491 \ldots$

As a further example of the use of (12)

$$K(3, 1, 1)=1+\frac{1}{8} K(8, 3, 1)$$

$$K(8, 3, 1)=3+\frac{1}{7} K(21, 8, 1)$$

$$K(21, 8, 1)=8+\frac{8}{55} K(55, 21, 1)$$

Thus

$$K(3, 1, 1)=1+\frac{3}{8}+\frac{1}{7}+\frac{1}{385} K(55, 21, 1)$$

$$=1{\cdot}6061 \ldots \text{ approx.}$$

Using (14), we get

$$K(10, 7, 1)=3{\cdot}5+20{\cdot}520+0{\cdot}204$$

$$=24{\cdot}224 \ldots \text{ approx.}$$

More simply, by (12) and (13)

$$K(10, 7, 1)=7+\frac{7}{99} K(99, 70, 7)$$

$$=24{\cdot}21 \ldots \text{ approx.}$$

When $c/(a-b)$ is very nearly unity we can use the formula

$$K(a, b, c)=\frac{ab}{a-b}\left\{\tfrac{1}{2} \log_\epsilon \frac{2ab}{(a-b)(a-b-c)}-\psi\left(\frac{a}{a-b}\right)\right\} \quad . \quad . \quad . \quad . \qquad (15)$$

where $\psi\left(\frac{a}{a-b}\right)$ is the logarithmic derivate of the gamma function, tables of the values of which are given in Russell's "Alternating Currents," Vol. I., p. 241.

For example, by (15)

$$K(8, 4, 3{\cdot}99)=27{\cdot}82 \ldots$$

It will be seen that the capacities of spherical condensers can be computed in all cases with little labour.

THE POTENTIALS AT THE INVERSE POINTS OF THE TWO SPHERES.

If P and Q be the inverse points of the two spheres in Fig. 2, then $AP \cdot AQ=a^2$ and $BP.BQ=b^2$. It easily follows that $AP=a\varepsilon^{-\alpha}=c+BP=c+b\varepsilon^{-\beta}$. From equation (4) we can readily find simple expressions for the potentials at P and Q due to the charge q. Equation (4) may be written

$$f_2\left(\frac{x-c}{b}\right)-\frac{ab}{a^2-cx}f_2\left\{\frac{x-c}{b}+\frac{c(x-a\varepsilon^{-\alpha})(x-a\varepsilon^{\alpha})}{b^2(a^2-cx)}\right\}=V.$$

Hence, if V_{P} be the potential at P due to the charge q, we have

$$V_{\mathrm{P}}\left\{1-\frac{ab}{a^2-ca\varepsilon^{-\alpha}}\right\}=V,$$

and thus by (8)

$$V_{\mathrm{P}}=V\frac{\varepsilon^{\omega}}{\varepsilon^{\omega}-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

By (1) we get

$$V_{\mathrm{Q}}=V\frac{\varepsilon^{-\alpha}}{\varepsilon^{\omega}-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If we denote the potentials at P and Q due to $-q$ by V'_{P} and V'_{Q}, we obviously have

$$V_{\mathrm{P}}+V'_{\mathrm{P}}=V, \quad \text{and} \quad V_{\mathrm{Q}}+V'_{\mathrm{Q}}=0.$$

Thus

$$V'_{\mathrm{P}}=-V\frac{1}{\varepsilon^{\omega}-1} \quad \text{and} \quad V'_{\mathrm{Q}}=-V\frac{\varepsilon^{-\alpha}}{\varepsilon^{\omega}-1}. \quad . \quad . \quad . \quad (18)$$

Note also that

$$\varepsilon^{\frac{\beta}{2}}=\left\{\frac{(s-b)(s-c)}{bc}\right\}^{\frac{1}{2}}+\left\{\frac{s(a-s)}{bc}\right\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and

$$2\frac{\varepsilon^{\omega}}{\varepsilon^{\omega}-1}=1+\left\{\frac{s(s-c)}{(a-s)(s-b)}\right\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

For instance, in the condenser $K(5, 2, 2)$, $BP=b\varepsilon^{-\beta}=\frac{1}{2}$, $BQ=8$, and if the potential of the inner sphere be V, $V_{\mathrm{P}}=2V$, $V_{\mathrm{Q}}=V/2$, $V'_{\mathrm{P}}=-V$, and $V'_{\mathrm{Q}}=-V/2$. By (12) also the charges on the inner and outer conductors are $3{\cdot}892\ V$ and $-3{\cdot}892\ V$ approximately.

Relations Between the Capacities of Spherical Condensers.

There are an infinite number of algebraical relations connecting the capacities of various condensers with one another. For example, provided that c is greater than 2, we can show by (9) and (7) that

$$K(c^2-1, c, 1)-cK(c^2-1, 1, c)=\frac{c^2-1}{c^2-2}K(c^2-2, 1, 1). \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Putting $c=4$, we get

$$K(15, 4, 1)-4K(15, 1, 4)=\frac{15}{14}K(14, 1, 1) \text{ exactly}=1{\cdot}154 \text{ (approx.)}.$$

THE FORCE ON THE INNER SPHERE OF A SPHERICAL CONDENSER.

If W denote the electrostatic energy stored in the condenser we have

$$W = q^2/2K = \tfrac{1}{2}KV^2.$$

As K always increases as the distance c between the centres of the spheres increases, we see that if the charge q be maintained constant, W diminishes as c increases, but that if V be maintained constant, W increases as c increases. We have now to find out whether the mutual force tends to increase or diminish c. When the charge is maintained constant it follows from the conservation of energy that the work done by the mutual force F must be supplied from the electrostatic energy. Hence F must act so that W diminishes and consequently c increases. When c is zero the equilibrium is unstable. It follows also that when the P.D. V is maintained constant c increases, and the electrostatic energy increases. Since

$$W_2 - W_1 = \int_1^2 F\delta c,$$

we see that when the P.D. is kept constant, the increase in the electrostatic energy $W_2 - W_1$ equals the concurrent work done $\int_1^2 F\delta c$. The battery which maintains the P.D. constant has therefore to supply an amount $2\,(W_2 - W_1)$ of energy.

If F denote the force tending to increase c we have $F = \frac{V^2}{2}\frac{\delta K}{\delta c}$, and thus by (10),

$$\begin{aligned} \frac{2F}{V^2} = {} & \Sigma \frac{n \cosh(\beta + n\omega)}{\sinh^2(\beta + n\omega)} \\ & - \frac{\cosh\alpha \cosh\beta}{\sinh\omega} \Sigma \frac{1}{\sinh(\beta + n\omega)} \\ & + \frac{\sinh\beta \cosh\alpha}{\sinh\omega} \Sigma \frac{\cosh(\beta + n\omega)}{\sinh^2(\beta + n\omega)} \qquad \text{(22)} \end{aligned}$$

the summation being taken from $n = 1$ to $n = \infty$.

It will be noticed that so long as V is constant the attractive force is the same whether the geometrical constants are a, b and c, or ma, mb and mc.

By substituting q/K for V in (22) we see that the force when q remains constant varies inversely as K^2, when the relative values of a, b and c, and therefore α and β, remain constant. Hence, for a given value of q the force when the constants are a, b and c would be m^2 times the force when the constants are ma, mb and mc.

When $c^2/(a-b)^2$ can be neglected compared with unity, we get, by (13)

$$F = \frac{\delta W}{\delta c} = \frac{a^2b^2c}{(a^3 - b^3)(a-b)^2}V^2 = \frac{c}{a^3 - b^3}q^2. \qquad \text{(23)}$$

Hence in this case the attractive force varies as the distance between the centres.

When the distance x $(= a - b - c)$ between the spheres is very small, we get (*l.c. ante*)

$$F = \frac{ab}{4(a-b)x}V^2. \qquad \text{(24)}$$

In this case, therefore, the force varies inversely as the distance between the spheres. If V be measured in electrostatic units (one electrostatic unit = 300 volts), F will be given in dynes. By means of (15) we can readily express F in terms of q^2, but the formula is not so simple.

EXTERNAL SPHERES.

Let us consider two conducting spheres (Fig. 3) with their centres at A and B. Let their radii be a and b, and let c be the distance between their centres. We shall denote their charges by Q_1 and Q_2, and their potentials by v_1 and v_2 respectively. We have, therefore,

$$Q_1=k_{11}v_1+k_{12}v_2 \text{ and } Q_2=k_{22}v_2+k_{12}v_1,$$

where k_{11} and k_{22} are the self capacity coefficients and k_{12} is the mutual capacity coefficient.

We shall first show how these coefficients can be computed in terms of a, b and c. To shorten the formulæ, we shall commence with the assumption that $v_1=1$ and $v_2=0$.

Let $f_1(x/a)$, where $AP=x$, denote the potential due to Q_1 at any point P on AB which is inside the sphere A. Similarly, let $f_2\{(c-x)/b\}$ be the potential due to Q_2 at the point P.

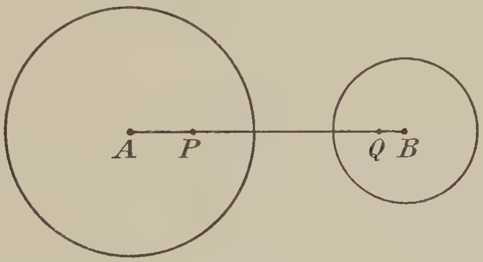

FIG. 3.

Then, if x lies between $+a$ and $-a$, we have

$$f_1(x/a)+\{b/(c-x)\}f_2\{b/(c-x)\}=1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (25)$$

Similarly, if x_1 lies between $c+b$ and $c-b$, we have

$$(a/x_1)f_1(a/x_1)+f_2\{(c-x_1)/b\}=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (26)$$

Now, writing

$$\frac{c-x_1}{b}=\frac{b}{c-x}$$

so that

$$\frac{a}{x_1}=\frac{a(c-x)}{c^2-b^2-cx}$$

and eliminating the second function between (25) and (26), we get

$$f_1\left(\frac{x}{a}\right)=1+\frac{ab}{c^2-b^2-cx}f_1\left\{\frac{a(c-x)}{c^2-b^2-cx}\right\}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (27)$$

At A, where $x=0$, we obviously have

$$f_1(0)=\frac{Q_1}{a}=\frac{k_{11}}{a}.$$

Thus

$$\frac{k_{11}}{a}=1+\frac{ab}{c^2-b^2}f_1\left(\frac{ac}{c^2-b^2}\right). \quad . \; . \; . \; . \; . \; . \; . \quad (28)$$

Let us introduce, as before, functions α, β and r, where r is the radius of the sphere which cuts A and B orthogonally, and

$$\sinh\alpha=\frac{r}{a},\quad \sinh\beta=\frac{r}{b}.\quad \text{We write also}$$

$$\sinh\omega=\sinh(\alpha+\beta)=\frac{cr}{ab},\ \text{and}$$

$$c=a\cosh\alpha+b\cosh\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

We find that (28) becomes

$$k_{11}\frac{\sinh\alpha}{r}=1+\frac{\sinh\alpha}{\sinh(\alpha+\omega)}f_1\left\{\frac{\sinh\omega}{\sinh(\alpha+\omega)}\right\}.$$

And by (27)

$$f_1\left\{\frac{\sinh\omega}{\sinh(\alpha+\omega)}\right\}=1+\frac{\sinh(\alpha+\omega)}{\sinh(\alpha+2\omega)}f_1\left\{\frac{\sinh 2\omega}{\sinh(\alpha+2\omega)}\right\}$$

Proceeding in this way, we get by substitution that

$$k_{11}=\sum_0^\infty\frac{r}{\sinh(\alpha+n\omega)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence, from symmetry

$$k_{22}=\sum_0^\infty\frac{r}{\sinh(\beta+n\omega)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

At the centre of the sphere B, the potential due to the charge Q_2 is Q_2/b, that is, k_{12}/b. Hence from (26), putting $x=c$, we get

$$f_1(0)=\frac{k_{12}}{b}=-\frac{a}{c}f_1\left(\frac{a}{c}\right)$$

Therefore

$$-k_{12}=\frac{r}{\sinh\omega}f_1\left(\frac{\sinh\beta}{\sinh\omega}\right)$$

And from (27)

$$f_1\left(\frac{\sinh\beta}{\sinh\omega}\right)=1+\frac{\sinh\omega}{\sinh 2\omega}f_1\left\{\frac{\sinh(\beta+\omega)}{\sinh 2\omega}\right\}$$

$$f_1\left\{\frac{\sinh(\beta+\omega)}{\sinh 2\omega}\right\}=1+\frac{\sinh 2\omega}{\sinh 3\omega}f_1\left\{\frac{\sinh(\beta+2\omega)}{\sinh 3\omega}\right\}$$

and so on. Hence

$$-k_{12}=\sum_1^\infty\frac{r}{\sinh n\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Comparing (30) with (10) we get at once

$$k_{11}=\frac{r}{\sinh\alpha}+K\left\{\frac{r}{\sinh\alpha},\ \frac{r}{\sinh(\alpha+\omega)},\ \frac{r\sinh\omega}{\sinh\alpha\sinh(\alpha+\omega)}\right\} \quad . \quad (33)$$

This may be written

$$k_{11}(a,b,c)=a+\frac{a^2}{c^2-b^2}K\left\{\frac{c^2-b^2}{a},b,c\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Similarly we find that

$$-k_{12}(a, b, c)=\frac{ab}{c}+\frac{a^2b^2}{c(c^2-a^2-b^2)}K\left\{\frac{c^2-a^2-b^2}{ab}, 1, 1\right\} \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

$$k_{22}(b, a, c)=b+\frac{b^2}{c^2-a^2}K\left\{\frac{c^2-a^2}{b}, a, c\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

The computation of the capacity coefficients, therefore, reduces to calculating the capacities of spherical condensers and so we can calculate them by the formulæ given above.

NUMERICAL EXAMPLES.

Poisson (*l.c. ante*) considers the case for which $a=1$, $b=3$ and $c=5$. By (34) we get

$$k_{11}(1, 3, 5)=1+\frac{1}{5^2-3^2}K(16, 3, 5)$$

and by (12) $$K(16, 3, 5)=3+\frac{3}{77}K(77, 16, 5)$$

If we only want a four-figure accuracy for k_{11} we can assume that

$$K(77, 16, 5)=K(77, 16, 0)=\frac{77.16}{77-16}$$

Hence $$k_{11}=1+\frac{3}{16}+\frac{3}{61}=1{\cdot}2367$$

Similarly, by (35), $$-k_{12}=\frac{3}{5}+\frac{3}{25}K(5, 1, 1)$$

$$=\frac{3}{5}+\frac{3}{25}+\frac{1}{200}K(24, 5, 1)$$

$$=\frac{3}{5}+\frac{3}{25}+\frac{3}{85}=0{\cdot}7516.$$

These numbers agree exactly with Poisson's calculations.

Similarly $k_{22}=3+\frac{3}{8}K(8, 1, 5)$

Proceeding as before, we get $k_{22}=3{\cdot}4719$.

As another example, take the system (1, 7, 10).

In this case $$k_{11}=1+\frac{1}{51}K(51, 7, 10)= \ . \ . \ .$$

Thus $$k_{11}=1{\cdot}160\ 112\ 4\ . \ . \ .$$

$$k_{22}=7{\cdot}576\ 596\ 7\ . \ . \ .$$

and $$-k_{12}=0{\cdot}814\ 326\ 5\ . \ . \ .$$

THE CAPACITY BETWEEN TWO SPHERES.

When the charges on the two spheres are equal but opposite in sign, the ratio of the positive charge to the potential difference between the spheres is called the capacity between the spheres. We shall denote it by $C\,(a, b, c)$ or by C. It is easy to show that

$$C(a, b, c)=\frac{k_{11}k_{22}-k^2_{12}}{k_{11}+k_{22}+2k_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

As a rule C is computed by first calculating the values of k_{11}, k_{22} and k_{12} and then substituting in (37). When, however, the spheres are equal to one another $k_{11}=k_{22}$ and (37) becomes

$$C(a, a, c)=\tfrac{1}{2}(k_{11}-k_{12}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

It readily follows from (34) and (35) that

$$C(1, 1, c)=\tfrac{1}{2}+\frac{1}{2c}\,K(c, 1, 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

For example, take the case of two spheres, the radius of each being 1 centimetre and the distance between thir centres being 4 cm. We have

$$\begin{aligned} C(1, 1, 4)&=\tfrac{1}{2}+\tfrac{1}{8}K(4, 1, 1) \\ &=\tfrac{1}{2}+\tfrac{1}{8}+\frac{1}{120}\,K(15, 4, 1) \\ &=\tfrac{1}{2}+\tfrac{1}{8}+\frac{1}{30}+\frac{1}{1680}\,K(56, 15, 1) \\ &=0{\cdot}670\;529\;88\;.\;.\;. \end{aligned}$$

SPHERES CLOSE TOGETHER.

When the spheres are so close together that it is laborious to apply (34) and (35) we can use the formulæ.*

$$k_{11}=\frac{r}{\omega}\left\{-\psi\left(\frac{\alpha}{\omega}\right)-\log_\epsilon\frac{\omega}{2}+\frac{\omega^2}{72}+\frac{7\omega^4}{43200}+\;.\;.\;\;-\frac{\alpha\beta}{12}-\frac{7\alpha^2\beta^2}{1440}-\;.\;.\;.\right\} \quad . \quad (40)$$

and
$$-k_{12}=\frac{r}{\omega}\left\{\gamma-\log_\epsilon\frac{\omega}{2}+\frac{\omega^2}{72}+\frac{7\omega^4}{43200}+\;.\;.\;.\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where $\psi(x)$ denotes the logarithmic derivate of the gamma function and γ is Euler's constant (0·577216). We have

$$\psi(x)+\gamma=-\frac{1}{x}+S_2x-S_3x^2+S_4x^3-\;.\;.\;.$$

where $S_m=\Sigma\frac{1}{n^m}$, the summation being taken from $n=1$ to $n=\infty$.

Let C_a and C_b denote the capacities of the two spheres when touching one another. In this case we have

$$C_a=k_{11}+k_{12}=-\frac{ab}{a+b}\left\{\psi\left(\frac{b}{a+b}\right)+\gamma\right\}=a\left\{1-\left(\frac{b}{a+b}\right)^2S_2+\left(\frac{b}{a+b}\right)^3S_3-\;.\;.\;.\;,\right\}$$

and
$$C_b=b\frac{ab}{(a+b)^2}\left\{S_2+\frac{b}{a+b}S_3+\frac{b^2}{(a+b)^2}S_4+\;.\;.\;.\right\}\dagger$$

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* Russell, Alternating Currents, Vol. 1, p. 240.

† J. J. Thomson, Electricity and Magnetism, Chap. 5.

It is to be noticed that when $b^3/(a+b)^3$ can be neglected compared with unity, we can write $C_a+C_b=a$. When the spheres are touching the following formulæ are exact :—

$$a=b, \quad C_a=a\log 2=0{\cdot}6931a=C_b$$

$$a=2b, \quad C_a=\frac{a}{6}\left(3\log 3+\frac{\pi}{\sqrt{3}}\right); \quad C_b=\frac{a}{6}\left(3\log 3-\frac{\pi}{\sqrt{3}}\right)$$

$$=0{\cdot}8516a \qquad\qquad =0{\cdot}2470a$$

$$a=3b, \quad C_a=\frac{a}{4}\left(3\log 2+\frac{\pi}{2}\right); \quad C_b=\frac{a}{4}\left(3\log 2-\frac{\pi}{2}\right)$$

$$=0{\cdot}9126a \qquad\qquad =0{\cdot}1272a$$

$$a=5b, \quad C_a=\frac{a}{12}(\log 2+3\log 6+\pi\sqrt{3}), \; C_b=\frac{a}{12}(\log 2+3\log 6-\pi\sqrt{3})$$

$$=0{\cdot}9592a \qquad\qquad =0{\cdot}0523a$$

When x/a and x/b can be neglected compared with unity, where $x=c-a-b$, we can show that

$$C_a+C_b=-\frac{ab}{a+b}\left\{\psi\left(\frac{a}{a+b}\right)+\psi\left(\frac{b}{a+b}\right)+2\gamma\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (42)$$

and

$$C=\frac{r}{\omega}\log_\epsilon\frac{2}{\omega}+\frac{r^2}{\omega^2}\frac{\psi\left(\frac{\alpha}{\omega}\right)\psi\left(\frac{\beta}{\omega}\right)-\gamma^2}{C_a+C_b} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (43)$$

$$=\frac{ab}{2(a+b)}\log_\epsilon\frac{2ab}{(a+b)x}+\frac{a^2b^2}{(a+b)^2}\cdot\frac{\psi\left(\frac{a}{a+b}\right)\psi\left(\frac{b}{a+b}\right)-\gamma^2}{C_a+C_b} \quad . \; . \quad (44)$$

where C is the capacity between the spheres.

For example, when $a=b$, we have, since $\psi(1/2)=-\gamma-2\log 2$ and $C_a+C_b=2a\log 2$

$$C=\frac{a}{4}\log\frac{a}{x}+\frac{a}{4}(2\gamma+2\log 2)$$

$$=\frac{a}{4}\left\{\log\frac{a}{x}+2{\cdot}540\,726\;.\;.\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (45)$$

When b/a is very small (44) becomes—

$$C=\frac{b}{2}\log\frac{2b}{x}+\gamma b. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (46)$$

This formula gives the capacity between a sphere of radius b and an infinite plane.

Hence, since $-k_{12}=C$ in this case, we get by (35)

$$K\left\{b, \frac{b^2}{2(b+x)}, \frac{b^2}{2(b+x)}\right\}=\frac{b}{2}\log\frac{2b}{x}+(\gamma-1)b, \quad . \; . \; . \; . \; . \; . \; . \quad (47)$$

when x is very small.

THE POTENTIALS AT THE INVERSE POINTS DUE TO Q_1 AND Q_2 RESPECTIVELY.

Looking back at (27) we see that the potential at the centre of the sphere A, where $x=o$, equals $1+$the potential at a point $x=a_1$, multiplied by $\sinh \alpha/\sinh(\alpha+\omega)$. Similarly the potential at the point $x=a_{n-1}$ equals $1+$the potential at the point $x=a_n$ multiplied by $\sinh(\alpha+\overline{n-1}\omega)/\sinh(\alpha+n\omega)$. We also see that $a_n=a\sinh n\omega/\sinh(\alpha+n\omega)$.

Hence, when n is very great

$$a_{n-1}=a_n=a\varepsilon^{-\alpha}=AP,$$

where P is the inverse point of the two spheres which is inside A. Hence, if V_{P} be the potential at this point due to Q_1, we have

$$V_{\mathrm{P}}=1+\varepsilon^{-\omega}V_{\mathrm{P}}, \text{ and so } V_{\mathrm{P}}=\varepsilon^{\omega}/(\varepsilon^{\omega}-1).$$

Hence, also $V_{\mathrm{Q}}=\varepsilon^{\beta}/(\varepsilon^{\omega}-1)$. If V'_{P}, V'_{Q} be the potentials due to Q_2, we see that

$$V'_{\mathrm{P}}=-1/(\varepsilon^{\omega}-1) \text{ and } V'_{\mathrm{Q}}=-\varepsilon^{\beta}/(\varepsilon^{\omega}-1).$$

We see, therefore, by superposition, that when the potentials of the spheres are V_1 and V_2, we have

$$(\varepsilon^{\omega}-1)V_{\mathrm{P}}=\varepsilon^{\alpha}(V_1\varepsilon^{\beta}-V_2)\,;\ (\varepsilon^{\omega}-1)V_{\mathrm{Q}}=V_1\varepsilon^{\beta}-V_2$$

$$(\varepsilon^{\omega}-1)V'_{\mathrm{P}}=V_2\varepsilon^{\alpha}-V_1\,;\ (\varepsilon^{\omega}-1)V'_{\mathrm{Q}}=\varepsilon^{\beta}(V_2\varepsilon^{\alpha}-V_1)$$

Hence, if $V_1/V_2=\varepsilon^{-\beta}$, $V_{\mathrm{P}}=V_{\mathrm{Q}}=o$; and $V'_{\mathrm{P}}=V_1$, $V'_{\mathrm{Q}}=V_2$.

ELECTRICAL IMAGES OF SPHERES.

If the surfaces of two spheres A and B are reflecting there will be a series of images formed inside A, and a similar series formed inside B. From this optical analogy Kelvin called the inverse spheres $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ where A and B are the centres of inversion (Fig. 4), the electrical images of the two spheres. It has to be noted that these electrical images do not coincide with the optical images. A_1 is the electrical image of B formed by A, and B_1 is the image of A formed by B. Similarly A_2 is the image of B_1 formed by A, and B_2 is the image of A_1 formed by B. In general A_n is the image of B_{n-1}, and B_n is the image of A_{n-1}.

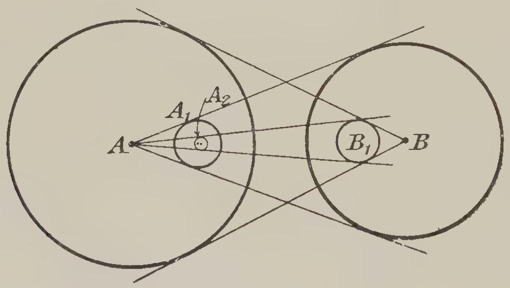

FIG. 4

It is not difficult to show that A_2 is the image (inverse sphere) of A formed by A_1, and in general that A_n is the image of A_{n-2} formed by A_{n-1}. If R_n be the radius of A_n and a_n the distance of its centre from A, it can be shown that

$$R_n=r/\sinh(\alpha+n\omega) \text{ and } a_n=a\sinh n\omega/\sinh(\alpha+n\omega).$$

Hence, when n is very large $a_n = a\varepsilon^{-\alpha}$ and the nth image practically coincides with the inverse point P.

By formula (33) we get

$$k_{11}(a, b, c) = a + K(a, R_1, a_1),$$

and by (12)

$$K(a, R_1, a_1) = R_1 + K(R_1, R_2, a_2 - a_1),$$

and

$$K(R_1, R_2, a_2 - a_1) = R_2 + K(R_2, R_3, a_3 - a_2), \text{ etc.}$$

Thus

$$k_{11}(a, b, c) = a + R_1 + R_2 + \ldots \qquad (48)$$

which agrees with (30).

Interpreting the equations we see that k_{11} equals the radius of A, together with the capacity of the spherical condenser formed by A and A_1. The capacity of this condenser equals the radius of A_1 plus the capacity of the condenser formed by A_1 and A_2, and so on. Hence the images divide the sphere A into an infinite number of spherical condensers, the capacity between any consecutive pair A_{n-1} and A_n being equal to the radius of A_n plus the capacity between the next pair A_n and A_{n+1}.

In the corresponding case of two cylinders it will be found that the capacities between the consecutive cylindrical images (inverse cylinders) are all equal to one another, and they also equal the capacity between the two cylinders.

It is interesting to note that the centres of the sphere A and of the images A_1, A_2, . . . coincide with the points where we have to place Kelvin's charges q_0, q_1, q_2, . . . so that these charges may produce a field external to the sphere which is identical with the field produced by the charge on A when its potential is unity, and that of B is zero. Kelvin's charges* also are given by $q_0 = a$, $q_1 = R_1$, $q_2 = R_2$, . . .

Both Poisson's and Kelvin's methods give the magnitudes and positions of the electrical image spheres.

ATTRACTIONS AND REPULSIONS OF TWO ELECTRIFIED SPHERES.

Let W denote the electrostatic energy of the system. Then

$$W = \tfrac{1}{2}k_{11}v_1^2 + \tfrac{1}{2}k_{22}v_2^2 + k_{12}v_1v_2 \qquad (49)$$

This can also be written in the form

$$W = \tfrac{1}{2}p_{11}q_1^2 + \tfrac{1}{2}p_{22}q_2^2 + p_{12}q_1q_2 \qquad (50)$$

where

$$p_{11} = k_{22}/\Delta^2, \quad p_{22} = k_{11}/\Delta^2, \quad p_{12} = -k_{12}/\Delta^2 \text{ and } \Delta^2 = k_{11}k_{22} - k_{12}^2 \qquad (51)$$

If we suppose the spheres to move, the rate at which the potential energy increases is given by

$$\frac{\delta W}{\delta c} = \frac{1}{2}\frac{\delta k_{11}}{\delta c}v_1^2 + \frac{1}{2}\frac{\delta k_{22}}{\delta c}\cdot v_2^2 + \frac{\delta k_{12}}{\delta c}v_1v_2 = nv_1v_2 - \tfrac{1}{2}lv_1^2 - \tfrac{1}{2}mv_2^2 \qquad (52)$$

Where $l = -\delta k_{11}/\delta c$, $m = -\delta k_{22}/\delta c$ and $n = \delta k_{12}/\delta c$, and l, m and n are positive quantities. The value of $\delta W/\delta c$ gives the force acting between the two spheres.

Similarly we find from (50) that when q_1 and q_2 are kept constant,

$$\frac{\delta W}{\delta c} = \frac{1}{2}\frac{\delta p_{11}}{\delta c}q_1^2 + \frac{1}{2}\frac{\delta p_{22}}{\delta c}q_2^2 + \frac{\delta p_{12}}{\delta c}q_1q_2.$$

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* Russell, Alternating Currents, Vol. 1, p. 238.

Substituting for p_{11}, p_{22} and p_{12} their values from (51), and writing for q_1 and q_2, $k_{11}v_1+k_{12}v_2$ and $k_{22}v_2+k_{12}v_1$, we readily find that

$$\frac{\delta W}{\delta c}=-(nv_1v_2-\tfrac{1}{2}lv_1^2-\tfrac{1}{2}mv_2^2)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Hence

$$\frac{\delta W}{\delta c}\,(v_1,\, v_2 \text{ constant})=-\frac{\delta W}{\delta c}\,(q_1,\, q_2 \text{ constant})\quad . \quad . \quad . \quad . \quad (54)$$

By the conservation of energy we see that when q_1 and q_2 are constant, the work done by the mutual force between them in moving the spheres must be taken from the electrostatic energy. Hence the spheres move so as to diminish the electrostatic energy. It follows also from (54) that when their potentials are maintained constant the spheres move so that the electrostatic energy is increased.

We see from (52) and (53) that $\delta W/\delta c$ vanishes when

$$\frac{v_1}{v_2}=\frac{n}{l}\pm\frac{\sqrt{n^2-lm}}{l}$$

and

$$\frac{q_1}{q_2}=\frac{k_{11}(v_1/v_2)+k_{12}}{k_{22}+k_{12}(v_1/v_2)}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

When for a given value of c either of these conditions is satisfied, the electrostatic energy at this distance is a maximum when the charges are kept constant, and a minimum when the potentials are kept constant. In order that (55) be satisfied it is necessary that v_1 and v_2 have the same sign. If they have opposite signs the electrostatic energy continually diminishes as the spheres approach one another when the charges are kept constant and continually increases when the potentials are maintained constant.

When the charges and therefore the potentials are of unlike sign, the mutual force is always attractive. When, however, they are of like sign the force is in general attractive when they are close together and repulsive when they are far apart. In an intermediate position determined by (55) the mutual force is zero.

SPHERES CLOSE TOGETHER.

When the spheres are at potentials ν_1 and ν_2, and the distance x between them is small compared with the radius of either, the force F between them is given by

$$\begin{aligned}\mathrm{F}=&-\frac{(\nu_1-\nu_2)^2ab}{4x(a+b)}\left\{1+\frac{(a^2+b^2-ab)x}{3ab(a+b)}\log\frac{(a+b)x}{2ab}+\frac{(2a^2+2b^2+ab)x}{18ab(a+b)}\right\}\\&-\frac{a^2+b^2-ab}{6(a+b)^2}\left\{\psi\left(\frac{b}{a+b}\right)\nu_1^2+\psi\left(\frac{a}{a+b}\right)\nu_2^2+2\gamma\nu_1\nu_2\right\}\\&-\frac{ab(a-b)}{6(a+b)^3}\left\{\psi'\left(\frac{b}{a+b}\right)\nu_1^2-\psi'\left(\frac{a}{a+b}\right)\nu_2^2\right\}\\&-\frac{ab}{6(a+b)^2}\nu_1\nu_2\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)\end{aligned}$$

When F is negative, the force is attractive, and when F is positive it is repulsive. It will be seen that when x is very small we only have repulsion when ν_1 is very nearly equal to ν_2. In all other cases the force is attractive.

If $\nu_1=\nu_2=\nu$, the force is repulsive. When $a=b$, $F=0{\cdot}07386\nu^2$; when $a=3b$, $F=0{\cdot}0430\nu^2$; and when a is very large compared with b, the force vanishes.

When $a=b$, (56) reduces to

$$F=-\frac{a(\nu_1-\nu_2)^2}{8x}\left\{1-\frac{12\gamma+1}{36}\cdot\frac{x}{a}-\frac{1}{6}\frac{x}{a}\log\frac{a}{x}\right\}+\frac{\log 2-\frac{1}{4}}{12}(\nu_1{}^2+\nu_2{}^2) \quad . \quad . \quad (57)^*$$

When $\nu_1=\nu_2=\nu$, we get

$$F=0{\cdot}073858\nu^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)\dagger$$

a formula that was given by Kelvin.

We see, however, that when x is very small the slightest disparity between the values of ν_1 and ν_2 will make the force attractive.

EQUAL SPHERES.

When the distance between two equal spheres is greater than the radius of either, the force F between them is given very approximately by

$$F=\left\{1+\frac{14}{y^6}+\frac{54}{y^8}\right\}\frac{q_1q_2}{c^2}-\left\{\frac{2}{y^3}+\frac{3}{y^5}+\frac{4}{y^7}+\frac{25}{y^3(y^2-1)^3}+\frac{75}{y^5(y^2-1)^3}\right\}\frac{q_1{}^2+q_2{}^2}{c^2} \quad . \quad . \quad (59)\ddagger$$

where $y=c/a$.

It is also given by

$$\mathrm{F}=\left\{\frac{1}{y^2-4}-\frac{1}{y^4}-\frac{1}{y^6}-\frac{1}{y^8-4y^5}\right\}\nu_1\nu_2-\left\{\frac{1}{y(y^2-4)}-\frac{1}{y(y^3-2y)^2}\right\}(\nu_1{}^2+\nu_2{}^2) \quad . \quad (60)\ddagger$$

When F is negative the force is attractive, and when F is positive it is repulsive. For example, when $c=4a$, the formulæ become

$$F=1{\cdot}004\,\frac{q_1q_2}{c^2}-0{\cdot}03456\,\frac{(q_1{}^2+q_2{}^2)}{c^2}$$

and $$F=0{\cdot}0792\nu_1\nu_2-0{\cdot}0208(\nu_1{}^2+\nu_2{}^2).$$

In this case F vanishes when $\nu_1/\nu_2=0{\cdot}283$, which is the same condition as $q_1/q_2=0{\cdot}0344$. When c/a is large, $F=\frac{q_1q_2}{c^2}-\frac{2a^3}{c^5}(q_1{}^2+q_2{}^2)$, and $F=\frac{a^2}{c^2}\nu_1\nu_2-\frac{a^3}{c^3}(\nu_1{}^2+\nu_2{}^2)$.

In using the formulæ given above it has to be remembered that when the pressure v and the charge q are given in electrostatic units the force F is given in dynes.

The electrostatic unit of pressure equals 300 volts and the electrostatic unit of charge (quantity) equals $1/(3\times10^9)$ coulombs.

ANALOGOUS PROBLEMS.

The formulæ given above can be usefully applied in the theory of heat, in electric and magnetic conduction, and in the theory of hydrodynamics. Consider, for instance, two spheres deeply embedded in a thermally homogeneous medium and maintained at constant temperatures θ_1 and θ_2. Let Q_1 and Q_2 denote the flow of heat per second from each sphere. When the flow has become steady, we get

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* Proc. Roy. Soc., Vol. 82, p. 531.

† Reprint, p. 83.

‡ Journal Inst. of El. Eng., Vol. 48, p. 257

$$Q_1=\frac{4\pi}{\sigma}(k_{11}\theta_1+k_{12}\theta_2) \text{ and } Q_2=\frac{4\pi}{\sigma}(k_{22}\theta_2+k_{12}\theta_1),$$

where σ is the thermal resistivity, and the values of k_{11}, k_{22} and k_{12} can be computed from the formulæ given above. In the particular case when $Q_1=-Q_2=Q$, we have

$$\frac{Q}{\theta_1-\theta_2}=\frac{4\pi}{\sigma}K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

where K is the electrostatic capacity between the two spheres.

Similarly, in electric conduction, we have, with the usual notation

$$I_1=\frac{4\pi}{\rho}(k_{11}V_1+k_{12}V_2)\ ;\ I_2=\frac{4\pi}{\rho}(k_{22}V_1+k_{12}V_1)$$

and when $I_1=-I_2=I$ we get

$$\frac{I}{V_1-V_2}=\frac{4\pi}{\rho}K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)^*$$

where ρ is the electric resistivity.

We see that when V_1 and V_2 are maintained constant and are of the same sign, the power supplied to the medium in which the spheres are immersed has, in general, a minimum value for a definite value of c.

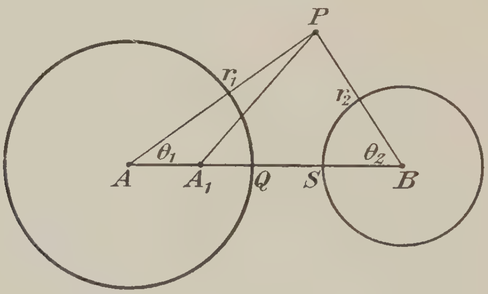

FIG. 5.

THE POTENTIAL AT ANY EXTERNAL POINT.

Let the point P (Fig. 5) be at distances r_1 and r_2 from A and B and let the angles PAB, PBA be θ_1 and θ_2 respectively. Let the potentials of the spheres be V_1 and 0 respectively. So far as the electric field outside the two spheres is concerned, we can replace the charges on the spheres by point charges q_0, q_1, q_2 . . . at A, A_1, A_2 . . . and q'_1, q'_2 . . . at B_1, B_2 . . . where

$$q_n=aV_1\frac{\sinh \alpha}{\sinh (n\omega+\alpha)},\ q'_n=-aV_1\frac{\sinh \alpha}{\sinh n\omega}$$

$$AA_n=a_n=a\frac{\sinh n\omega}{\sinh (n\omega+\alpha)} \text{ and } BB_n=b_n=b\frac{\sinh \{(n-1)\omega+\alpha\}}{\sinh n\omega}$$

---

* Jeans, Electricity and Magnetism, p. 351.

Thus the potential V at the point P due to the given charges is got by

$$
\begin{aligned}
V &= \frac{q_0}{r_1} + \frac{q_1}{(r_1^2 - 2r_1a_1\cos\theta_1 + a_1^2)^{\frac{1}{2}}} \\
&\quad + \frac{q_2}{(r_1^2 - 2r_1a_2\cos\theta_1 + a_2^2)^{\frac{1}{2}}} + \ldots \\
&\quad + \frac{q'_1}{(r_2^2 - 2r_2b_1\cos\theta_2 + b_1^2)} + \ldots \\
&= \frac{k_{11}V_1}{r_1} + \frac{k_{12}V_1}{r_2} \\
&\quad + \frac{a^2V_1P_1(\cos\theta_1)}{r_1^2}\sum_1^\infty \frac{\sinh\alpha\sinh n\omega}{\sinh^2(n\omega+\alpha)} \\
&\quad + \frac{a^3V_1P_2(\cos\theta_1)}{r_1^3}\sum_1^\infty \frac{\sinh\alpha\sinh^2 n\omega}{\sinh^3(n\omega+\alpha)} + \ldots \\
&\quad - \frac{b^2V_1P_1(\cos\theta_2)}{r_2^2}\sum_1^\infty \frac{\sinh\alpha\sinh\overline{(n-1\,\omega+\alpha)}}{\sinh^2 n\omega} - \ldots \quad (61)
\end{aligned}
$$

where $P_1(\cos\theta_1)$, $P_2(\cos\theta_2)$ . . . are zonal harmonics.

If the second sphere be at potential V_2, we have to add to the series (61) a similar series which can be written down by symmetry at once.

The Maximum Value of the Electric Stress.

It is important in practice to know the maximum values of the electric stress at points on the surface of the sphere. These values obviously occur at the points Q and S (Fig. 5). Denoting the maximum value of the stress at Q by $R_{\text{max.}}$, we get*

$$
\begin{aligned}
R_{\text{max.}} = \frac{\cosh^2(\alpha/2)}{a\sinh(\alpha/2)}\Bigg[V_1\left\{\frac{\sinh(\alpha/2)}{\cosh^2(\alpha/2)} + \frac{\sinh(\alpha/2+\omega)}{\cosh^2(\alpha/2+\omega)} + \ldots\right\} \\
-V_2\left\{\frac{\sinh(\omega-\alpha/2)}{\cosh^2(\omega-\alpha/2)} + \frac{\sinh(2\omega-\alpha/2)}{\cosh^2(2\omega-\alpha/2)} + \ldots\right\}\Bigg] \quad . \quad . \quad (62)
\end{aligned}
$$

The value of the stress at S is got by writing V_2 for V_1, b for a and β for α in (62).

When the spheres are equal, the maximum value of the electric stress between them is given by

$$R_{\text{max.}} = \frac{V_1 - V_2}{x}f_1 + \frac{2V_2}{x}(f_1 - f) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

where $$f = \sinh\alpha\cosh(\alpha/2)\left\{\frac{\sinh(\alpha/2)}{\cosh^2(\alpha/2)} + \frac{\sinh(3\alpha/2)}{\cosh^2(3\alpha/2)} + \ldots\right\}$$

$$f_1 = 2\sinh\alpha\cosh(\alpha/2)\left\{\frac{\sinh(\alpha/2)}{\cosh^2(\alpha/2)} + \frac{\sinh(5\alpha/2)}{\cosh^2(5\alpha/2)} + \ldots\right\}$$

and $\cosh\alpha = 1 + x/(2a)$.

Simplified formulæ for computing f and f_1 are given in the Journ. of the Inst. of El. Engin., Vol. LVII., p. 223.

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* Russell, Alternating Currents, Vol. 1, p. 253.

It is also shown that for a given pair of spherical electrodes the value of $R_{\text{max.}}$ at which a disruptive discharge takes place is constant for distances between the spheres varying from about 1 mm. up to a distance equal to three times the diameter of either electrode. The author has shown that the value of $R_{\text{max.}}$ at 25°C. and 76 cm. pressure for electrodes the diameters of which vary in size from half a centimetre to 50 cm. is given by

$$R_{\text{max.}}=27\cdot4+\frac{14\cdot1}{\sqrt{a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

when the potentials of the electrodes are equal and opposite at the instant of discharge; a is the radius of an electrode in centimetres and $R_{\text{max.}}$ is given in kilovolts per centimetre. The maximum inaccuracy of (64) over the range mentioned is approximately 1 per cent.

When $V_1=-V_2=E/2$, formula (63) becomes

$$R_{\text{max.}}=\frac{E}{x}f$$

and thus

$$E=\left(27\cdot4+\frac{14\cdot1}{\sqrt{a}}\right)\frac{x}{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

If we fix the value of x it is easy to see from this formula that in certain cases E has a maximum value for a given value of a. For instance, if x is 1 cm., then E has the greatest value when $a=2\cdot53$ cm., that is, approximately 1 in. Taking the experimental values of E from the Paper referred to above, we get :—

Radius of each electrode in cm.	0·25	0·5	1	3·125	6·25	12·5	25
Disruptive pressure in kilovolts	20·8	26·7	30·5	31·9	31·5	30·6	29·8

The results of the first three cases were given by A. Heydweiller* and the last four are taken from the standardisation Rules of the American Institution of Electrical Engineers.

THE MAXIMUM POSSIBLE ATTRACTION BETWEEN SPHERES IN AIR.

Taking Heydweiller's results for spheres 1 cm. in diameter, we readily find that the attractions between the electrodes at the instant of discharge are given by the following table :—

x in cm.	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
E in kilovolts...	8·37	11·37	14·55	17·31	19·92	22·05	24·09	25·59	27·00
F in dynes	193·1	219·0	250·0	264·6	274·5	271·7	268·7	256·3	244·4

The maximum attraction occurs when x equals $1\cdot2a$ approximately.

Let a pair of spherical electrodes each have a radius a and be at a distance x apart. Let F_1 be the attraction between them when the disruptive voltage is E_1. The potentials are supposed equal and opposite at the instant of discharge. Consider also a pair of electrodes the radii of which are each equal to na, and let the distance between them be nx. Let F_2 be the force between them at the instant of the disruptive discharge. Then $F_1=\alpha E^2_1$ and $E_1=(27{\cdot}4+14{\cdot}1/\sqrt{a})(x/f)$, where α and f depend only on the ratio x/a. It readily follows that

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* Entladungspotentiale Ann. der Phys. und Chemie, Vol. 48, p. 235.

$$\frac{F_2}{F_1}=\left\{n\frac{27{\cdot}4+14{\cdot}1/\sqrt{na}}{27{\cdot}4+14{\cdot}1/\sqrt{a}}\right\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

Hence this ratio is independent of x and so if x' be the distance between the first pair of electrodes which gives to F_1 a maximum value, F_2 will have a maximum value when the distance between the spheres in the second case is nx'.

We saw above that when $a=\frac{1}{2}$ cm. approx. F has its maximum value when $x'=0{\cdot}6$ cm. Let us take $n=10$, then if F_2 be the attraction for electrodes of 5 cm. radius at the instant of discharge $F_2=50{\cdot}8F_1$, and so the maximum value of F_2 is nearly 13940 dynes.

An electrostatic balance therefore having spherical electrodes each ten centimetres in diameter and 5 centimetres apart would be very suitable for measuring potential differences up to 100,000 volts.

DISCUSSION.

Sir **Oliver Lodge** (communicated): I feel sure that Dr. Russell's Paper is a valuable contribution to knowledge; and I admire the way in which he has tackled the problem of the two electrified spheres. An electrostatic balance with spherical electrodes, instead of flat plates, would have many advantages, provided always that the surroundings did not exercise too much disturbance, and provided dust particles or irregularities on the spheres did not promote an undesired kind of brush discharge accompanied by air currents.

Mr. T. **Smith** congratulated the President on a valuable piece of work. Having cleared up a subject at which physicists had been working for so long, the author would have been entitled to employ as his unit of length a characteristic wave-length of the most widely distributed element; which, as Clerk Maxwell had remarked, was the only unit fitted for those investigators whose work might outlast the earth! In regard to the use of "image" in an optical sense, in the course of the Paper, this hardly seemed the correct word to employ.

Dr. D. **Owen** said that in investigating the attraction between conductors in the case where the disruption point was being reached, it had to be remembered that ions were present, which would interfere with the distribution of the electric field. This lent some uncertainty to practical methods based on this method for measuring voltages; the time factor entered into this question. The same remark applied to the estimation of voltage in terms of the distance between spheres at which sparking just occurred.

The **Author**, in reply, said that it might be more correct to speak of the "inverse" of one sphere with respect to the other. The equation (64) to which Dr. Owen referred was an empirical one. Possibly the presence of ions might account for this formula, as theoretically the value of $R_{\max}$ should be constant.

III. *The Homographic Treatment of the Symmetrical Optical Instrument.* By G. Temple, *Birkbeck College.*

Received May 23, 1922.
(Communicated by Dr. D. Owen.)

ABSTRACT.

This Paper gives a homographic treatment of the perfect, symmetrical optical instrument; discusses its focal planes, principal planes and nodal points; and shows that these latter are sufficient to define uniquely the optical properties of the instrument.

The object of this Paper is to replace the rather cumbrous algebra of Abbe's* theory of the symmetrical optical instrument by the more elegant methods of modern geometry.

Definition of the Instrument.

The perfect, symmetrical, optical instrument is defined by two conditions—the condition for stigmatism, and the condition for axial symmetry.

1. *Stigmatism.*—The effect of the instrument must be to form a point image of a point object. Now, to any ray through the point object, there corresponds a ray through the point image. Hence, rays coplanar in the object space correspond to rays coplanar in the image space. Hence the instrument sets up a homographic, or (1, 1) correspondence between the points, lines and planes of the object and image spaces.†

2. *Axial Symmetry.*—Associated with the instrument there must be a certain line—the optic axis—such that to rays in a plane π containing the axis, there correspond rays in the same plane; and to lines perpendicular to the axis, there correspond lines also perpendicular to the axis. Hence to points on the axis there correspond points on the axis, so that the axis is a self-corresponding or "double" line. Also since all lines perpendicular to the axis in the plane π meet in the same point W at infinity, W is a double point. Now the instrument sets up a (1, 1) correspondence between the points or the lines of the plane π. Therefore the plane π contains three double points, the vertices of a triangle whose sides are three double lines. Hence the other two double points lie on the axis and are either both real or both imaginary.

Focal Planes.

The line at infinity, w, of any plane perpendicular to the axis contains all the double points, W, and is therefore itself a double line. Now the focal planes φ, φ, of the image and object space are defined as the planes corresponding to the plane at infinity ω regarded as belonging to the object or image space respectively. But the double line w lies on ω. Therefore it also lies on φ, φ'. Hence the focal planes φ, φ' are perpendicular to the axis.

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* Drude: "The Theory of Optics," pp. 14-28 (1902). v. Rohr: "The Formation of Images in Optical Instruments," pp. 83-124 (1920). Gapski: "Theorie des Optischen Instrumente pack Abbe" (1894).

† Jessopp: "The Line Complex."

PRINCIPAL PLANES AND NODAL POINTS.

These are introduced so as to exhibit their duality.

Principal Planes.	*Nodal Points.*
Consider a pair of corresponding planes η, η' whose intersection is the line w.	Consider a pair of corresponding points N, N' whose join is the axis.
There is a (1, 1) correspondence between the lines on η, η' and the common line w is self-corresponding.	There is a (1, 1) correspondence between the lines through N, N', and the common line the axis is self-corresponding.
Hence corresponding lines on η, η' are in perspective.	Hence corresponding lines through N, N' are in perspective.
When the centre of perspective is at infinity η, η' are principal planes.	When the axis of perspective is at infinity N, N' are nodal points.
Then the join of corresponding points of η, η' meets the axis at infinity.	Then the intersection of corresponding lines through N, N' is at infinity.
If μ, μ' be another pair of principal planes, any line l meeting the axis at infinity, is met by η, η', μ, μ' in two pairs of corresponding points. Hence l is a double line.	If P, P' is another pair of nodal points any point L at infinity is joined to N, N', P, P' by two pairs of corresponding lines. Hence L is a double point.

But this is only true for telescopic systems of unit magnifying power. Hence in general there is only one pair of principal planes or nodal points.

CONSTRUCTION OF IMAGE.

Let A, A' be corresponding points; and let AN, $A'N'$ meet φ, φ' in F, F', and η, η' in H, H'.

If Ω' is the point at infinity on $A'N'$, since A, A'; N, N'; F, Ω'; H, H' are pairs of corresponding points, $A'N'$ is parallel to AN and $[ANFH]=[A'N'\Omega'H]$. (using the notation of Mobius for cross-ratios). Hence, φ, φ'; η, η', N, N' suffice to determine A' corresponding to A.

MIRROR SYSTEMS.

The distinction between mirror systems and lens systems is that whereas in a lens system, a collineation is effected between the points, lines and planes of two different spaces; in a mirror system, the collineation is effected between the points, lines and planes of the same space.

Hence in a mirror system, the corresponding pairs of points on the axis form an involution, of which the pole of the mirror P is evidently one double point. Hence the other double point Q is also real, and the centre of the involution (*i.e.*, the point corresponding to the point at infinity, or the focus) is midway between P and Q; and any two corresponding points are harmonically separated by P and Q. The principal points obviously coincide at P, and the nodal points at Q.

DISCUSSION.

Mr. T. **Smith** congratulated the author on the brevity with which his method covered ground that is usually treated at great length, especially in German textbooks. The importance of the perfect instrument lies not in its realisation in practice, which is unattainable except for the plane mirror, but in that it is the ideal to which actual instruments must approximate. In practice perfection is attainable with respect to a single plane object, but perfection with regard to more than one plane implies complete perfection. Clerk Maxwell employed a method independent of constructional details and applied it to the practical case where the principal and nodal points coincide ; this case might also be of interest to the author. Clerk Maxwell further introduced one physical property, *viz.*, the ratio of the velocities of light in the object and image spaces, and deduced hence the properties of the focal lengths. In practical cases there is a (1, 1) correspondence between rays, but not between points, and this aspect of the matter has been elegantly treated by the method of characteristic functions, notably by Hamilton.

Mr. F. J. W. **Whipple** said that the Paper was extremely concise, and might be amplified with advantage. In presenting it the author had stated that correspondence between points implies correspondence between planes. Would he give the proof of this theorem ?

The **Author**, in reply to Mr. Smith, said it had occurred to him that the consequences of a (1, 1) correspondence between rays might be deduced by means of the geometry of the line complex. The proof asked for by Mr. Whipple was as follows. Let A, B, C, D be four coplanar points in the object space and A', B', C', D' four corresponding points in the image space. Then the latter four points are also coplanar. For let AD meet BC in X. Since X lies on both AD and BC, the corresponding point X' must lie on both $A'D'$ and $B'C'$. Hence these must be coplanar.

ADDENDUM BY THE AUTHOR.

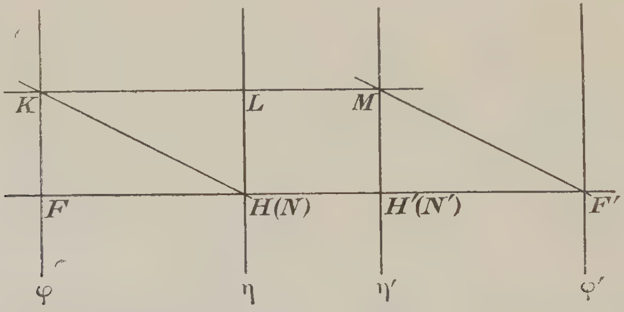
The following example bears upon Mr. T. Smith's remark as to focal lengths :—

Let the axis of the instrument meet the focal planes φ, φ' in F, F' ; and the principal planes η, η' in H, H'.

Then the geometrical expression of the fact that the refractive indices of the material of those parts of the object and image spaces outside the instrument, are the same, is that H, H' respectively coincide with the nodal points N, N'.

Let K be any point on φ. Draw KLM parallel to the axis to meet η, η' in L, M.

Now F' corresponds to the point at infinity on the axis, which is the same as the point at infinity on KLM.



Hence the ray KL passes through F', and since L,M are corresponding points, it also passes through M.

Hence the ray KL corresponds to the ray MF'.

Hence the point K corresponds to the point at infinity on MF'.

Join KH, *i.e.*, KN. Then K also lies at infinity on a line through H' (N') parallel to KH (N). Therefore KH, MF' are parallel.

But $KF=MH'$.

Therefore $FH=H'F'$, *i.e.*, the focal lengths are numerically equal.

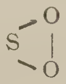
IV. *On the Structure of the Sulphur Dioxide Molecule.*

By A. O. Rankine, *D.Sc., Professor of Physics, and* C. J. Smith, *M.Sc., A.R.C.S., D.I.C., Research Student, Imperial College of Science and Technology.*

Received July 15, 1922.

ABSTRACT.

This Paper deals with the question of the arrangement of the atoms in the molecule of gaseous sulphur dioxide and the manner in which they are linked together. Two suggested arrangements are considered in detail, and evidence is brought forward, based on the determination of molecular dimensions as derived from viscosity measurements, which suggests that the more probable arrangement is that in which chemical bonds exist between *each* pair of the three atoms, corresponding to the formula



According to the views put forward by Langmuir* sulphur dioxide is one of the numerous examples of chemical compounds in which the mode of linking together of the atoms is by electron sharing. The completed molecule is conceived to be an arrangement in which each oxygen atom, by holding electrons in common with the other constituents, has attained the configuration appropriate to a neon atom; and the sulphur atom, by a like process, has simulated an atom of argon. The extent of electron sharing is determined by the total deficiency of electrons as compared with the corresponding inert atoms. Thus there is a total deficiency of six, two on account of each oxygen atom, the atomic number of which is 8, as compared with the 10 of neon, and two on account of the sulphur atom, the atomic number being in this case 16, as compared with the 18 of argon. This requires that in the sulphur dioxide molecule six electrons altogether must be held in common by the constituent atoms taken in pairs; and we have to inquire into the probable distribution of the electron sharing, since it can take place in more ways than one.

Langmuir, although he does not give a diagram of the electron arrangement, represents the molecule by the symbols

$$O{=}S{-}O$$

The significance of the single bond is that two electrons are shared; the double bond signifies the sharing of four electrons; thus the necessary total of six electrons are held in common. This arrangement demands, however, that the two oxygen atoms should be on opposite sides of the sulphur atom. It also raises the puzzling question as to why the singly bound oxygen atom attaches itself at all, in view of the apparently saturated condition of the hypothetical molecule $O{=}S$. For present purposes we will not wait to attempt to answer this question, but will examine an alternative possible arrangement of the sulphur dioxide molecule which is worthy of consideration. It has, we believe, already been suggested—we are not sure whence the suggestion originally comes. It is a triangular arrangement which is represented thus

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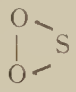
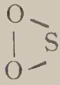
* I. Langmuir, Journ. Amer. Chem. Soc., Vol. 41, p. 868 (1919).



From the point of view of electron sharing this would signify that two electrons were shared at each link—or six altogether—so that the corresponding inert atom electron arrangements are approximately attained. There must, of course, be some distortion, on account of the sulphur atom being larger in size than those of oxygen; but account cannot be taken of it, owing to insufficient knowledge of its character.

The recent measurements of one of us (C.J.S.)* of the mean collision area of the sulphur dioxide molecule in the gaseous state permit the production of evidence which helps us to estimate the alternative probabilities of one or the other of the two above molecular structures being the true one. This evidence, in so far as it is valid, decides very definitely in favour of the second alternative, and the purpose of the present Paper is to show how this comes to be the case.

As has already been pointed out, there are good grounds for believing that oxygen atoms in combination resemble neon atoms in size and shape, and that sulphur atoms in like manner correspond to argon atoms. Dimensionally, therefore, it may be assumed that the sulphur dioxide molecule will have the properties of a particular arrangement of the two neon atoms and one argon atom. If a possible arrangement is chosen we can calculate the mean area which the model so constituted would present in collision for all orientations, and compare the result with the actual mean collision area deduced from determinations of the viscosity of the gas. The general application of this device has already been described,† and need be repeated here only so far as the special case demands. The sizes of the collision spheres of argon and neon are known from viscosity measurements—their radii are respectively 1·44 Å and 1·15 Å. The distances between the centres of atoms in combination are also known from the crystal measurements of W. L. Bragg.‡ For the kind of chemical combination operative in SO_2 this distance is the sum of the radii of the outer electron shells of the atoms in question. Between sulphur and oxygen this distance is 1·68 Å; between two oxygens it is 1·30 Å.

It will be more convenient to deal first with the model corresponding to the structure , for the reason that it is quite determinate—a consideration which, as will be seen later, does not apply to the model representing O=S—O. The centres of the collision spheres—or the nuclei of the atoms—lie at the corners of an isosceles triangle, the equal sides of which are 1·68 Å long, and the base 1·30 Å. The centre of the argon sphere (corresponding to the sulphur atom) is the apex, and the neon (oxygen) centres are at the other corners. Fig. 1 represents the arrangement drawn to scale, all three centres, or nuclei, being in the plane of the diagram.

This is the model of which we have to find the mean collision area for all possible orientations. It is one which does not lend itself to analytical treatment of a reasonably simple character, on account of the absence of an axis of complete symmetry; and we have therefore dealt with the problem partially by graphical methods. The axis SO' is chosen, and the orthogonal projection in a direction making an angle θ with SO' is drawn. This consists, of course, of three overlapping circles with centres at the corners of the triangle into which the triangle $SO'O''$ projects. In finding the mean projection area for a fixed value of θ it is necessary to contemplate all the equally probable positions of O'' with respect to the axis SO', *i.e.*, we must imagine O'' describing a circle round SO', and find the projections for suitably chosen positions of O'' on this circle. The projections of S and O' in any particular case remain fixed, and the projection of O'' describes the ellipse into which the above-mentioned circle projects. The necessary co-ordinates are readily found, and the corresponding circular projections can be drawn and measured. Owing to partial symmetry it is only necessary to examine the results of positions of O'' on half its

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* C. J. Smith, Phil. Mag., Vol. 44, p. 508 (1922).

† A. O. Rankine, Proc. Phys. Soc., Vol. 33, p. 362.

‡ W. L. Bragg, Phil. Mag., Vol. 40, p. 169 (1920).

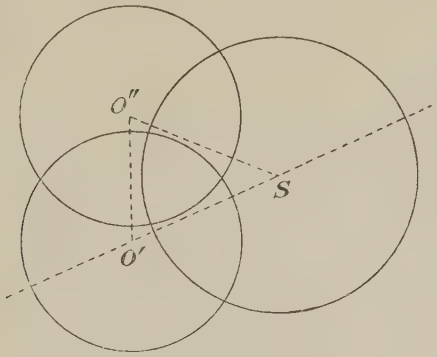

FIG. 1.—POSSIBLE ARRANGEMENT OF ATOMS IN MOLECULE.

$O'O'' \equiv 1{\cdot}30$ Å; $SO' = SO'' \equiv 1{\cdot}68$ Å.

Radii.—Large sphere $\equiv 1{\cdot}44$ Å; small spheres $\equiv 1{\cdot}15$ Å.

circular path around SO', and the corresponding angle from 0 to π has been divided into the intervals $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$ in order to attain a sufficient accuracy in estimating the mean area of projection. The general form of projection for a fixed value of θ is two unvarying overlapping circles corresponding to S and O', and the circle of varying position corresponding to O''. In each position it was only necessary to measure with the planimeter that part of the latter circular projection which was not eclipsed by the other circles; for formulæ already exist* for dealing analytically with the mean projection of S and O' alone. By drawing the appropriate graph and integrating mechanically the additional mean area $\bar{A}$ for projection in the θ direction—additional, that is to say, to the calculable value due to S and O' alone—was determined. Values of $\bar{A}$ were thus obtained for several directions of projection with respect to SO'; actually the values of θ taken were

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* A. O. Rankine, Proc. Phys. Soc., Vol. 33. p. 362.

$0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$, and $\frac{\pi}{2}$. It has already been shown by one of us (A.O.R.)* that if $\bar{\bar{A}}$ is the final mean area of projection for all orientations in space which are equally probable,

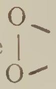
$$\bar{\bar{A}} = \int_0^{\pi/2} \bar{A} \sin\theta \,.\, d\theta.$$

The values of $\bar{A}$ above mentioned were therefore multiplied by the corresponding values of $\sin\theta$, and plotted against θ; and the integral was evaluated mechanically. The result obtained is

$$\bar{\bar{A}} = 0{\cdot}127 \times 10^{-15} \text{ cm.}^2.$$

To this there has to be added, in order to obtain the total mean projection area of our model, the mean area of projection corresponding to S and O', obtained by calculation from the formula already mentioned. This proves to be

$$0{\cdot}865 \times 10^{-15} \text{cm.}^2,$$

so that it may be stated that the model corresponding to the structure  for sulphur dioxide has a mean collision area

$$0{\cdot}99 \times 10^{-15} \text{cm.}^2.$$

It is estimated that the accuracy of this calculation, having regard to the limitations of the planimeter measurements of area, is within one per cent. The actual mean collision area of sulphur dioxide, as derived from viscosity measurements is

$$0{\cdot}94 \times 10^{-15} \text{cm.}^2,$$

and this is sufficiently near to that of the model to enable the difference to be attributed to experimental errors and inexactitude in our assumptions.

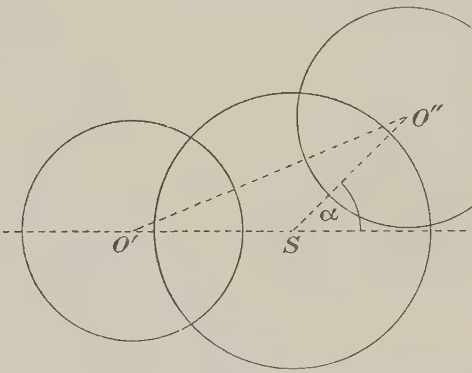


FIG. 2.—POSSIBLE ARRANGEMENT OF ATOMS IN MOLECULE.
$SO' = SO'' = 1{\cdot}68$ Å.
Radii.—Large sphere $= 1{\cdot}44$ Å; small spheres $= 1{\cdot}15$ Å.

We now proceed to consider the other model, namely, that corresponding to the structure suggested by Langmuir, O=S—O. Here, to a certain extent, the model is indeterminate; but fortunately it is possible to dismiss it without formal calculation, for reasons which will appear. According to what has already been said, this model may be represented by the full lines in Fig. 2, where the centres are, as before, in the plane of the diagram, and the spheres have the same sizes. SO' and SO'' are the same as before but, since it is no longer supposed that the two oxygens share electrons between themselves, the distance $O'O''$ is not known, nor is the angle α. Following Langmuir's usual method of representation α would be 45°, and the corresponding electron arrangement necessitates its being less than 90°; but there is no need to insist upon this particular value for the purpose of the following argument. It is certain that the mean area of projection of the model (which could be determined, were α known, in precisely the same way as that already indicated) would diminish as the angle α increased. For if α is so small that the plane touching the spheres O' and S does not cut the sphere O'', there is no possibility, in any orientation, of the spheres O' and O'' eclipsing one another except in regions covered by the sphere S. This means that the mean area of projection is the area of the circle with centre S plus two equal and entirely independent contributions arising from the excrescences O' and O''. This proves to be

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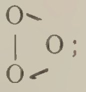
* A. O. Rankine, Proc. Roy. Soc., A. Vol. 98, p. 360.

$$1{\cdot}08 \times 10^{-15}\text{cm.}^2.$$

Preliminary drawings of projections, taking $\alpha=45°$ made it manifest that the effect —a reduction of course—of the mutual eclipsing of O' and O'' independently of S was very small, since it occurs only at orientations of infrequent occurrence. Even if the angle is greater than 45°, the mean projection area does not become reduced to $0{\cdot}99 \times 10^{-15}\text{cm.}^2$—that of the former model considered—until $O'O''$ becomes the 1·30 Å appropriate to the sharing of electrons between the two oxygens; and this as we have seen, still gives a value somewhat in excess of the experimentally determined mean collision area of the sulphur dioxide molecule.

The calculations which we have made are thus quite definitely in favour of the view that in the sulphur dioxide molecule there is a single bond, *i.e.*, two electrons shared, between each pair of the three constituent atoms, as compared with that which regards electron linkage as absent between the oxygen atoms. It would be interesting to know how far sulphur dioxide is equivalent in this respect to ozone, which is frequently represented as



but there seems little hope of obtaining the necessary experimental data, owing to the instability of this gas.

This opportunity may be taken of mentioning that the graphical method described in this Paper has been applied to the molecules COS and CS_2 for which an approximate analytical treatment has been given by one of us (A. O. R.).* The values of the mean collision area thus estimated are :—

COS	$1{\cdot}09 \times 10^{-15}\text{cm.}^2$,
CS_2	$1{\cdot}31 \times 10^{-15}\text{cm.}^2$.

The former compares satisfactorily with that deduced from viscosity measurements, namely, $1{\cdot}06 \times 10^{-15}\text{cm.}^2$; the latter cannot yet be compared with experimental results.

* A. O. Rankine, Phil. Mag., Vol. 44, p. 292 (1922).

DISCUSSION.

Dr. D. OWEN inquired whether the results obtained could be interpreted as favouring either the static or the dynamic theory of the atom.

Prof. RANKINE replied in the negative, no deductions to which the test could be applied having been made by the advocates of the dynamic theory. Prof. Bohr had told him that his theory led to a definite mechanism of combination for fluorine agreeing with the static mechanism of the salt type, in which atoms combine in virtue of their respective excess and defect of electrons as compared with the inert-gas atoms of their periodic group. The method described in the Paper could not be applied to such compounds as they are not volatile. Compounds comprising two non-metals combined by the sharing of electrons, as in the present case, had not yet been brought within the scope of Bohr's theory.

Dr. E. H. RAYNER referred to the thermal expansion of crystals and inquired whether the ordinary coefficient of expansion applied to the distances between the centres of the atoms which had been assumed in Fig. 1 of the Paper. In that case, was it justifiable to apply these distances to atoms in the gaseous state ?

Prof. RANKINE replied that X-ray measurements were supporting the view that the ordinary coefficients of expansion are applicable to atomic distances in crystals, but the Lewis-Langmuir theory implies that even in a gas the atoms which are sharing electrons are contiguous. This assumption leads to consistent results.

V. *The Thermal Effect of Vapours on Rubber. By* A. S. HOUGHTON, *B.Sc., A.I.C.*

RECEIVED JULY 1, 1922.

(COMMUNICATED BY SIDNEY SKINNER, M.A., F.INST.P.)

ABSTRACT.

This Paper relates to the heating of rubber when suddenly immersed in various vapours. It describes the method of measuring the thermal changes by means of thermo-electric junctions and gives the results obtained with benzene, pyridine, ammonia and water-vapour. The effect is compared to that of water-vapour on cellulose.

DURING the course of an investigation on the surface action of rubber the phenomenon of the passage of gas through rubber membranes was frequently encountered, and it seemed important to consider this matter in greater detail. It will be remembered that Graham found that the passage of gases through rubber was not according to the ordinary laws of porous plug diffusion, and that his explanation was that the gas was liquefied on entering the rubber, passed through the rubber as a liquid and was evaporated back to its normal state on leaving the rubber on the opposite side. The modern idea considers the first operation as some form of adsorption, and consequently we expect some thermal changes on the membrane during its exposure to the gas.

EXPERIMENTAL CONSIDERATIONS.

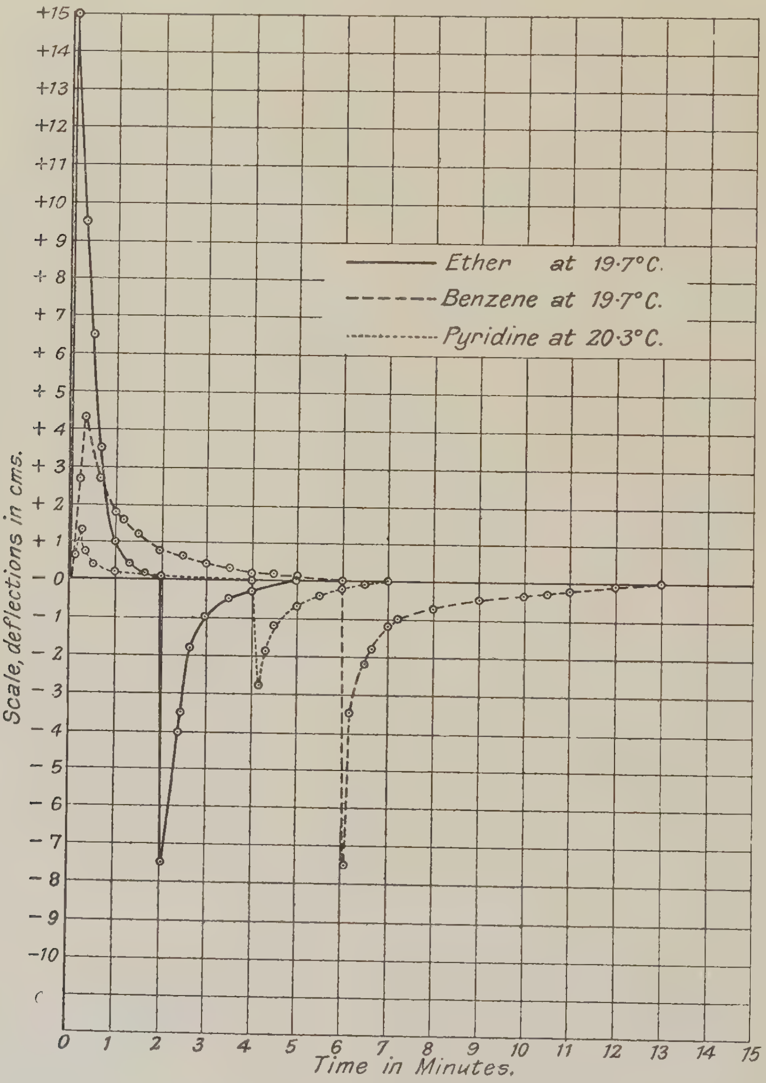
Preliminary experiments were first made with two exactly similar delicate thermometers, the bulb of one being covered with a thin film of rubber by means of rubber solution, whilst the other was left bare. When these were plunged into carbon dioxide it was noticed that the rubbered thermometer recorded a higher temperature than the other, thus indicating that some heating effect was going on in the rubber.

A more accurate means of recording this heating effect was afforded by thermo-electric junctions of copper and Eureka wire. Lengths of these wires were soldered together and the alternate junctions coated with rubber solution. The heating effect was measured by means of a delicate low-resistance mirror galvanometer to which these junctions were connected.

It was shown by experiments using this method that a definite heating effect was given by various vapours such as chloroform, benzene, toluene, pyridine and essential oils such as oil of lavender and oil of guaniol. It was also proved that if alternate unrubbered junctions were placed in such vapours no heating effect was given, so that the effect was due to some heating of the rubber film itself.

The final form of apparatus consisted of a brass jacket with a wooden lid, through a hole in which the junctions were placed and held in position by a packing of cotton-wool. In this way the junctions were protected from convection currents of air, thus ensuring a steady galvanometer deflection, since the whole could be lifted over and placed in the bottle containing the vapour.

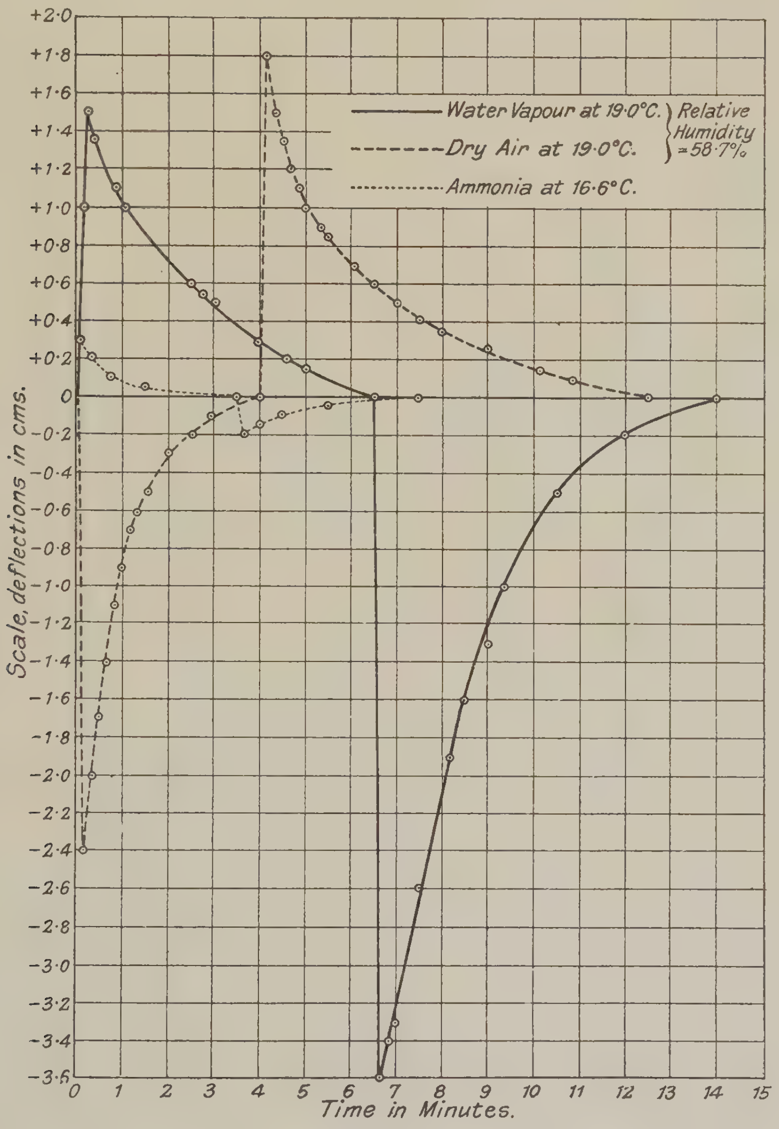
Experiments were carried out in the following manner: The liquid in the bottle was well shaken, to ensure that the space above it was saturated with vapour, and the jacket with the junctions placed over it. The movement of the beam of light from the mirror galvanometer was noted, times being taken with a stop-watch until the beam returned to zero. Then the jacket with the junctions was taken away from the bottle of vapour, and placed over nut-charcoal, so that any traces of vapour remaining in the jacket were rapidly absorbed. The deflections of the beam to the opposite side of the scale produced by this withdrawal were noted as before. These deflections were plotted against time and specimen curves for ether,



GRAPH 1.

benzene and pyridine are shown in Graph 1. These curves have been drawn to scale and consequently it has been thought unnecessary to give tables of observations.

As shown in Graph 1, the type of curve obtained in these experiments is quite a regular one. The chief points to be noted are the sudden heating on plunging the junctions into the vapour followed by a gradual return to zero, as the rubber becomes saturated with the vapour, and the sudden cooling on withdrawing the



Graph 2.

junctions followed by another gradual return to zero as the vapour evaporates from the surface of the vapour.

The next step in the investigation was to see if such gases as ammonia and carbon dioxide, which pass rapidly through rubber membranes, had a similar heating effect on the rubber-coated junctions. The pure dry gases were contained in wide-necked stoppered bottles and were allowed to stand for some time at room temperature. The experiments were carried out in the usual way, and although the deflections produced were much smaller they showed that the behaviour of carbon dioxide and ammonia was quite normal. In the case of ammonia junctions of platinum and iron were used since the gas was found to attack the copper junctions, thus giving faulty results. The curve for ammonia is given in Graph 2, and we can see it is identical in type with the previous curves.

It was thought interesting in view of recent work on hygrometry to try the effect of moist and dry atmospheres. A small wide-necked bottle was lined inside with three or four layers of wet filter-paper and the usual set of junctions were slipped into the moist air in the bottle. A heating effect was at once given and the behaviour was similar in every way to that of the vapours previously tried. The dry atmosphere was obtained by means of a little concentrated sulphuric acid at the bottom of a similar bottle, and in this case a cooling effect was noted on introducing the rubbered junctions and a heating effect on their withdrawal. The curves for these two experiments are shown in Graph 2.

Finally, a calibration of junctions was carried out so that some idea of the temperature value of the beam deflections could be obtained, and it was found that a deflection of 14 cm. approximately indicated 1°C. difference in temperature between the alternate junctions.

THEORETICAL CONSIDERATIONS.

The only previous work with any bearing on this research is that done on the heating of cellulose by water-vapour, and references to various work on this subject are given at the end of the Paper.

The heating of cellulose by water-vapour is generally attributed to the condensation of the aqueous vapour on the cellulose fibres with consequent liberation of heat of condensation. The results obtained with moist and dry atmospheres in the latter part of this work certainly point to some similarity in the behaviour of rubber and cellulose. Rubber differs from cellulose, however, in its non-fibrous structure and wide range of action with vapours. So it is probable that only part of the heat produced is due to the condensation of the vapour on the rubber, and the additional heat is supplied either by some form of chemical union, or by the formation of a solid solution between the vapour and rubber. The loose nature of this second phase is shown by the rapid return to the normal state on withdrawal of the rubber from the vapour, but the experimental facts undoubtedly prove the existence of this phase.

In conclusion, I desire to thank my supervisor, the Principal of the Chelsea Polytechnic, for suggesting the subject of this research, and also the Department of Scientific and Industrial Research for the grant that has enabled me to carry out this work.

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DISCUSSION.

Principal S. Skinner said it was interesting to see to what a large number of vapours india-rubber responds, as if it could smell them! In addition to the vapours mentioned in the Paper, the author had tried a number of those which form the basis of artificial scents, with marked effect. It might be worth considering whether the sense of smell is not connected with the adsorption of vapours by membranes covering the olfactory nerve-endings. The apparatus constitutes a sort of hygrometer for vapours other than water, and the humidity of the atmosphere is indicated by measurements of the kind illustrated in Fig. 2. The investigation began from the phenomenon of the diffusion of vapours through a rubber membrane, and possibly some Fellows present had experience in this connexion during the war.

Mr. **J. S. Anderson** asked whether it would be feasible to weigh the adsorbed vapour and see whether there was any time lag between adsorption and rise of temperature. Such a measurement would help to decide between chemical action and adsorption as an explanation of the phenomenon. The latter explanation is favoured by the relative heating effects of ether and benzene, which are roughly in the ratio of 3 : 1, for in the case of charcoal the heating effects are in the opposite direction, but are in the same ratio.

Dr. **H. Borns** asked whether the author had experimented with water and other substances in the liquid state. It is a fact that rubber swells in water and most other liquids, sometimes apparently without any chemical action taking place. It would be interesting to know whether the same effect can be observed with gases.

Mr. **F. J. W. Whipple** said it had been proposed to saturate the fabric of pilot balloons with ammonia for 24 hours before inflation to enable it to expand without rupture or the development of "pin-holes." Could the author give any information bearing on this suggestion? If it is the case that water evaporating from rubber gives the same effect as from cotton, the former fabric might have advantages over the latter for the purposes of hygrometry.

Dr. **D. Owen** said that it was most interesting to see the well-known phenomenon connected with cellulose and water vapour extended to quite different substances; but quantitative measurements of the calories liberated would be necessary to decide between adsorption and chemical action as an explanation of what occurred. The observations of rise of temperature need to be supplemented by a knowledge of the heat capacity and emissivity of the rubber, &c. An attempt might be made to measure the gain in weight, and so relate the heat evolved to the latent heat of the vapour—if such connection exists. The rubber membrane has the great advantage over cellulose that its area can be measured, so that the thickness of the adsorbed film might be estimated.

Dr. **J. S. G. Thomas** said that during the war it was sometimes found that balloons which had been filled with coal-gas could not be re-inflated after deflation. This was apparently due to chemical action of the gas on the rubber, which made the latter sticky.

Author's reply (communicated) :—Although I appreciate the helpful nature of the discussion on my Paper it is rather difficult to give definite answers to many of the questions raised. I should, however, like to emphasise one or two points with regard to the action described in my Paper. In the first place the action of the vapour on the rubber is practically instantaneous, the maximum heating effect being reached in many cases within 5 seconds. The return to the normal state, on withdrawing the rubber from the vapour, is equally rapid, so that any weighing of the amount of vapour absorbed would have to be done in an atmosphere saturated with that vapour. Another point is that the amount of rubber on the junctions is small, being less than

0·1 gms. The final decision between the chemical action and adsorption therein would, as Dr. Owen pointed out, depend entirely on a quantitative estimation of the amounts of vapour absorbed and heat evolved, and so some delicate work will have to be carried out before this point is settled.

With regard to the questions concerning balloon fabric it is already a well-known fact that rubber keeps much better in the presence of a slight trace of vapours such as carbon bisulphide and it is probable that ammonia acts in a similar way. It may be that rubber saturated with such a vapour is less prone to oxidation, which is one of the chief causes of the deterioration of rubber ; in addition, the vapour slightly warms the rubber and thereby tends to increase its elastic properties. The case of balloons filled with coal-gas is an extreme one. Here we have the hydrocarbon vapours, which coal-gas contains, in close contact with the rubber for some considerable period. As my experiments show, these vapours have a much greater effect than such gases as ammonia, and consequently the action, whether chemical or one of adsorption, is intensified, and the surface of the rubber is attacked. In my own experiments it has been noticed that if the rubber is used for some time with the more powerful vapours it becomes sticky, but no marked change in volume has been observed.

A DEMONSTRATION of a Calling Device for Wireless Telegraphy was given by T. D. PARKIN, *of Marconi's Wireless Telegraph Co., Ltd.*

THE device is intended for ships employing a single wireless operator, and is adapted to ring an alarm bell on receipt of waves having a characteristic group frequency in addition to a characteristic high frequency. The transmitter comprises a brass balance-wheel carrying a diametrally arranged bar of soft iron and moving between fixed field magnets, the circuit of which is closed periodically by contacts carried by the wheel. The connections are such that the oscillation of the wheel is maintained, and a second contact moving therewith effects the emission of wave-groups of the required group-frequency. The receiver comprises a field-magnet to the coils of which regular pulses of current are imparted on receipt of the above-mentioned wave-groups, acting upon a balance-wheel the rim of which consists of a ring of steel so magnetised as to have poles at opposite ends of a diameter. The natural frequency of this balance-wheel is adjusted to that of the expected wave-groups, so that after about 11 of these have been received the oscillations of the wheel attain sufficient amplitude to effect the ringing of an electric alarm bell provided with a hold-on circuit. The apparatus is mounted on springs and gimbals.

DISCUSSION.

Capt. C. W. HUME inquired whether provision had not been made for an " S.O.S." signal.

Mr. R. L. SMITH-ROSE inquired as to the range of the apparatus.

Dr. A. O. RANKINE inquired whether the receiver might not be affected by a letter comprising several dots, such as " H," sent at the proper group frequency.

Mr. PARKIN, in reply, stated that an adaptation to " S.O.S." signals was being considered, but had not yet been perfected. The apparatus had been worked at a range of 110 miles. It was less sensitive than a human operator, but would be effective at all distances at which distress signals could be of any value. The receiver should be adjusted so as not to respond to fewer than 11 impulses, in order to be immune from disturbance by dot letters or by the "erase" sign.

A DEMONSTRATION of an Apparatus for Testing the Tensile Strength of Gas Mantles was given by Mr. J. T. ROBIN.

THE apparatus comprises a vertical cylindrical float which is surrounded by water in a cylindrical container. It is surmounted by a cup carrying wax, and the lower edge of the mantle to be tested is attached to the float by allowing the melted wax to solidify while the mantle dips into it, the latter being suspended in the usual way. The mantle is then put in tension by running off the water slowly, so that the mantle takes a gradually increasing proportion of the weight of the float. As soon as the mantle ruptures the float drops and automatically effects cessation of the flow of the water. The apparatus having been previously calibrated, the breaking weight can be read from the height of the remaining water, which is shown by an inspection tube connected to the container at its lower end, and backed by an adjustable scale. Special precautions are observed in handling the mantle and burning off the cellulose. The shear strength is taken to be proportional to the tensile strength. It is found that some mantles can support 500 times their own weight.

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